Inverted oscillator: pseudo hermiticity and coherent states

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It is known that the standard and the inverted harmonic oscillator are different. Replacing thus ω by $\pm i\omega$ in the regular oscillator is necessary going to give the inverted oscillator H^r. This replacement would lead to anti- $\mathcal{PT}$-symmetric harmonic oscillator Hamiltonian $(\mp iH^{os})$. The pseudo-hermiticity relation has been used here to relate the anti-$\mathcal{PT}$-symmetric harmonic Hamiltonian to the inverted oscillator. By using a simple algebra, we introduce the ladder operators describing the inverted harmonic oscillator to reproduce the analytical solutions.We construct the inverted coherent states which minimize the quantum mechanical uncertainty between the position and the momentum.
This paper is dedicated to the memory of Omar Djemli and Nouredinne Mebarki who died due to covid 19.

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1. Introduction

The inverted oscillator, equipped with a potential exerting a repulsive force on a particle, has been widely studied [1-18]. Such system can be completely solved as the standard harmonic oscillator whose properties are well known.

However, the physics of the inverted harmonic oscillator is different, because its energy spectrum is continuous and its eigenstates are no longer square integrable.The inverted oscillator can be applied to various physical systems such as [1,19-21], the tunneling effects, the mechanism of matter-wave bright solitons, the cosmological model, and the quantum theory of measurement.

In fact, the predominant idea in the literature is that the inverted oscillator is obtainable from the harmonic oscillator by the replacement $\omega \rightarrow \pm i\omega$. Of course, in spite of many useful analogies, it is important to know that the two oscillators (harmonic and inverted) reveal different characteristics. In other words, the inverted oscillator generates a wave packet which are not square integrable and there is no zero-point energy. In comparison with the harmonic oscillator, the physical applications of the inverted harmonic oscillators are limited, since their Hamiltonian is parabolic and the eigenstates are scattering states. The analytic continuation of angular velocity $\omega \rightarrow \pm i\omega$ performs a transformation of a non-Hermitian harmonic oscillator $(\mp iH^{os})$ to inverted one H^r.

In general, non-Hermitian Hamiltonians have been used to describe several physical dissipative systems. Such Hamiltonians do not cause a legitimate probabilistic interpretation due to the shortage of the unitarity condition in their corresponding quantum description. In non-Hermitian quantum mechanics it, was found that the criteria for a quantum Hamiltonian to have a real spectrum is that it possesses an unbroken $\mathcal{PT}$ symmetry ($\mathcal{P}$ is the space-reflection operator or parity operator, and $\mathcal{T}$ is the time-reversal operator) [22,23]. The concept of $\mathcal{PT}$-symmetry has found applications in several areas of physics. Once the non-Hermitian Hamiltonian H is invariant under the combined action of $\mathcal{PT}$ (*i.e.* H commutes with $\mathcal{PT}$) and its eigenvectors are also those of the $\mathcal{PT}$ operator, then the energy eigenvalues E of the system are real and in this case the $\mathcal{PT}$-symmetry is unbroken.

An alternative approach to explore the basic structure responsible for the reality of the spectrum of a non- Hermitian Hamiltonian is by the notion of the pseudo-hermiticity introduced in Ref. [24]. An operator H is said to be pseudo-Hermitian if

$$H^{\dagger} = \eta H \eta^{-1}, \tag{1}$$

where the metric operator

$$\eta = \rho^{\dagger}\rho,\ \eta^{-1} = \left(\rho^{\dagger}\rho\right)^{-1}, \tag{2}$$

is a linear, invertible and Hermitian operator, we say that the Hamiltonian is pseudo-Hermitian or quasi-Hermitian if it satisfies the relation (1).

The pseudo-Hermiticity allows to link the pseudo-Hermitian Hamiltonian H with an equivalent Hermitian Hamiltonian h

$$h = \rho H \rho^{-1}, \tag{3}$$

where the operator ρ called Dyson operator is linear and invertible. Due to the energy spectrum of $(\pm iH^{os})$ being completely imaginary, we notice that $(\mp iH^{os})$ is anti-$\mathcal{PT}$-symmetric *i.e.*

$$\mathcal{PT}(\pm iH^{os})\mathcal{PT} = (\mp iH^{os}). \tag{4}$$

We recall that a $\mathcal{PT}$-symmetric system can be transformed to an anti-$\mathcal{PT}$-symmetric one by replacing $H^{os} \rightarrow$

$(\pm iH^{os})$ [25-28], which changes the physical structure of the system. In other words, a Hamiltonian H is said to be anti-$\mathcal{PT}$-symmetric if it anticommutes with the $\mathcal{PT}$ operator $\{\mathcal{PT}, H\} = 0$. In analogy with the $\mathcal{PT}$-symmetric case, we call the anti-$\mathcal{PT}$-symmetry of Hamiltonian H unbroken if all of the eigenfunctions of H are eigenfunctions of $\mathcal{PT}$, *i.e.* when the energy spectrum of H is entirely imaginary $E = iE^*$ [29].

In this paper, we generate from the anti-$\mathcal{PT}$-symmetric Hamiltonian $(\pm iH^{os})$ an inverted Hermitian harmonic oscillator-type H^r and also its solution. In Sec. 2, we recall briefly some properties of the standard harmonic and inverted oscillators In Sec. 3, introducing an appropriate quantum metric, we link the anti-$\mathcal{PT}$-symmetric Hamiltonian $(\pm iH^{os})$ to the inverted oscillator Hamiltonian H^r.This procedure allows us to obtain the pseudo-ladder operators, the set of solutions and also to define the full orthonormalization relation of the eigenstates for inverted harmonic oscillator H^r. In Sec. 4, using the pseudo-ladder operators, we will address the problem constructing of coherent states associated to inverted oscillator H^r. We obtain the mean values of the position and momentum operators in the evolved coherent states and furthermore we calculate the corresponding Heisenberg uncertainty. An outlook over the main results is given in the conclusion.

2. Summary of standard harmonic and the inverted oscillators

Let us recall briefly the ladder operator approach of the usual harmonic oscillator:

$$H^{os} = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 = \frac{\hbar\omega}{2}\left(a^+ a + a a^+\right), \quad (5)$$

where

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{p}{\sqrt{2m\hbar\omega}},$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - i\frac{p}{\sqrt{2m\hbar\omega}}, \quad (6)$$

The operators a and a^+ satisfying the commutation relation

$$\left[a, a^\dagger\right] = 1. \quad (7)$$

Were introduced to facilitate the solution of the eigenvalue problem. Eigenstates of (5) in Fock space are the Fock or number states $|n\rangle^{os}$ with the eigenvalues $\omega\left(n + 1/2\right)$), where $a\left|n\right\rangle^{os} = \sqrt{n}\left|n-1\right\rangle^{os}$, $a^\dagger\left|n\right\rangle^{os} = \sqrt{n+1}\left|n+1\right\rangle^{os}$ and n is a non-negative integer.

We then have a nice mechanism for computing the eigenstates of the Hamiltonian, but we can also express expectation values using the raising and lowering operators. This leads to the useful representation of x and p:

$$x = \sqrt{\frac{\hbar}{2\omega m}}\left(a^\dagger + a\right), \qquad p = i\sqrt{\frac{\hbar\omega m}{2}}\left(a^\dagger - a\right), \quad (8)$$

such that, we can compute any arbitrary expectation values that depend upon these quantities, merely by knowing the effects of the raising and lowering operators upon the eigenstates of the Hamiltonian.

From this, we can evaluate that the energy eigenvalues

$$H^{os}\psi_n^{os}(x) = E_n\psi_n^{os}(x)$$
$$= \hbar\omega\left(n + \frac{1}{2}\right)\psi_n^{os}(x); \quad n \in \mathbb{N}, \quad (9)$$

and the normalized condition for the eigenfunctions is verified

$$\langle\psi_m^{os}\,|\psi_n^{os}\rangle = \delta_{mn}. \quad (10)$$

We see that the energy eigenvalues $E_0 = \hbar\omega/2$ of the ground state

$$\psi_0^{os}(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{\omega m}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left[-\frac{\omega m}{2\hbar}x^2\right], \quad (11)$$

is a very significant physical result because it tells us that the energy of a system described by a harmonic oscillator potential cannot have zero energy.

In contrast with the harmonic oscillator, the inverted oscillator has a Hamiltonian with the following form:

$$H^r = \frac{1}{2m}p^2 - \frac{1}{2}m\omega^2 x^2 = -\frac{\hbar\omega}{2}\left(a^{\dagger 2} + a^2\right). \quad (12)$$

The Hamiltonian (12) is formally obtainable from (5) by the replacement

$$\omega \to i\omega, \quad (13)$$

similarly, the case $(-i\omega)$ would serve equally well.

On the other hand, for an imaginary frequency, *i.e.* for the inverted harmonic oscillator, we get

$$a \to A = e^{i\frac{\pi}{4}}\left(\sqrt{\frac{m\omega}{2\hbar}}x + \frac{p}{\sqrt{2m\omega\hbar}}\right), \quad (14)$$

$$a^+ \to \bar{A} = e^{i\frac{\pi}{4}}\left(\sqrt{\frac{m\omega}{2\hbar}}x - \frac{p}{\sqrt{2m\omega\hbar}}\right), \quad (15)$$

thus, the Hamiltonian (12) can take the following form

$$H^r = \frac{i\hbar\omega}{2}(\bar{A}A + A\bar{A}), \quad (16)$$

where the non-Hermitian pseudo-ladder operators $\left(A, \bar{A}\right)$ are characterized by $\left[A, \bar{A}\right] = 1$ in an analogous way to the ladder operator $\left(a, a^\dagger\right)$ for the harmonic oscillator.

Knowing that the eigenfunctions of the harmonic oscillator are normalized, we ask the question if the inverted oscillator eigenfunctions are also normalized? Clearly, they are not $\langle\psi_m^r\,|\psi_n^r\rangle \neq \delta_{mn}$. This can be seen when calculating the normalization condition for the pseudo-ground state $\psi_0^r(x)$ of the obtained inverted oscillator: from Eq. (11) by changing ω to $i\omega$

$$\psi_0^r(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{i\omega m}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left[-i\frac{\omega m}{2\hbar}x^2\right]. \quad (17)$$

One can easily verify that the normalization for this state diverges as follows:

$$\langle\psi_0^r\,|\psi_0^r\rangle = \int_{-\infty}^{+\infty}\psi_0^{*r}(x)\psi_0^r(x)dx = \frac{1}{2^n n!}\left(\frac{\omega m}{\pi\hbar}\right)^{\frac{1}{2}}\int_{-\infty}^{+\infty}dx \to \infty, \tag{18}$$

the reason for this divergence is that the substitution ω by $i\omega$ is unsuitable. we will remedy this inconsistency in what follows.

3. Pseudo-ladder operators in the inverted harmonic oscillator

The Hermitian Hamiltonian H^r and the non-Hermitian Hamiltonian (iH^{os}) are related by a formal replacement $\omega \to i\omega$. The challenge is to establish a consistent relation between the quantum mechanical formalism for the Hermitian Hamiltonian $H^r$and the non-Hermitian one (iH^{os}), we propose that instead of considering this formal transformation, we use the relation that it is valid for any self-adjoint operator, *i.e.* observable, in the Hermitian system to possess a counterpart in the non-Hermitian system given by

$$\rho^{-1}(iH^{os})\rho = H^r. \tag{19}$$

In order to connect the non-Hermitian harmonic oscillator Hamiltonian (iH^{os}) to the Hermitian inverted oscillator H^r, we perform a Dyson type transformation ρ such that [30]

$$\rho = \exp\left\{-2\left[\frac{\epsilon}{2}\left(a^\dagger a + \frac{1}{2}\right) + \mu_-\frac{a^2}{2} + \mu_+\frac{a^{\dagger 2}}{2}\right]\right\} = \exp\left[-\vartheta_-\frac{a^2}{2}\right]\exp\left[-\frac{\ln\vartheta_0}{2}\left(a^\dagger a + \frac{1}{2}\right)\right]\exp\left[-\vartheta_+\frac{a^{\dagger 2}}{2}\right], \tag{20}$$

and

$$\vartheta_+ = \frac{2\mu_+\sinh\theta}{\theta\cosh\theta - \epsilon\sinh\theta}, \qquad \vartheta_0 = \left(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\right)^{-2} = \mu_+\mu_- - \chi,$$
$$\vartheta_- = \frac{2\mu_-\sinh\theta}{\theta\cosh\theta - \epsilon\sinh\theta}, \qquad \chi = -\frac{\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta}{\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta}, \qquad \theta = \sqrt{\epsilon^2 - 4\mu_+\mu_-}, \tag{21}$$

where ϵ is a real parameter whereas μ_+ and μ_- are complex ones.

With the help of the following relations

$$\begin{cases} \exp\left[\vartheta_-\frac{a^2}{2}\right]\left(a^\dagger a + \frac{1}{2}\right)\exp\left[-\vartheta_-\frac{a^2}{2}\right] = \left(a^\dagger a + \frac{1}{2}\right) + \vartheta_- a^2 \\ \exp\left[\vartheta_+\frac{a^{\dagger 2}}{2}\right]\left(a^\dagger a + \frac{1}{2}\right)\exp\left[-\vartheta_+\frac{a^{\dagger 2}}{2}\right] = \left(a^\dagger a + \frac{1}{2}\right) - \vartheta_+ a^{\dagger 2} \end{cases}, \tag{22}$$

$$\begin{cases} \exp\left[\frac{\ln\vartheta_0}{2}\left(a^\dagger a + \frac{1}{2}\right)\right] a^2 \exp\left[-\frac{\ln\vartheta_0}{2}\left(a^\dagger a + \frac{1}{2}\right)\right] = \frac{a^2}{\vartheta_0} \\ \exp\left[\vartheta_+\frac{a^{\dagger 2}}{2}\right] a^2 \exp\left[-\vartheta_+\frac{a^{\dagger 2}}{2}\right] = a^2 - 2\vartheta_+\left(a^\dagger a + \frac{1}{2}\right) + \vartheta_+^2 a^{\dagger 2} \end{cases}, \tag{23}$$

$$\begin{cases} \exp\left[\frac{\ln\vartheta_0}{2}\left(a^\dagger a + \frac{1}{2}\right)\right] a^{\dagger 2} \exp\left[-\frac{\ln\vartheta_0}{2}\left(a^\dagger a + \frac{1}{2}\right)\right] = \vartheta_0 a^{\dagger 2} \\ \exp\left[\vartheta_-\frac{a^2}{2}\right] a^{\dagger 2} \exp\left[-\vartheta_-\frac{a^2}{2}\right] = a^{\dagger 2} + 2\vartheta_-\left(a^\dagger a + \frac{1}{2}\right) + \vartheta_-^2 a^2 \end{cases}, \tag{24}$$

we deduce, under the action of the operator ρ, the transformed Hamiltonian of the harmonic oscillator :

$$\rho^{-1}H^{os}\rho = \hbar\omega\rho^{-1}\left(a^\dagger a + \frac{1}{2}\right)\rho = \frac{\hbar\omega}{\vartheta_0}\left\{[\vartheta_0 - 2\vartheta_+\vartheta_-]\left(a^\dagger a + \frac{1}{2}\right) + \left[\vartheta_-\vartheta_+^2 - \vartheta_0\vartheta_+\right]a^{\dagger 2} + \vartheta_- a^2\right\}. \tag{25}$$

We notice that Eq. (25) and Eq. (12) have the same structure in their operator content provided that we impose on the parameters $(\vartheta_+, \vartheta_-,\ \vartheta_0)$ the following conditions

$$\vartheta_+ = -i,\ \vartheta_- = \frac{i}{2}, \vartheta_0 = 1, \tag{26}$$

from these constraints, the Dyson operator Eq. (20) takes now the simplified form[i]

$$\rho = \exp\left[-\frac{i}{4}a^2\right]\exp\left[\frac{i}{2}a^{\dagger 2}\right], \qquad \rho^{-1} = \exp\left[-\frac{i}{2}a^{\dagger 2}\right]\exp\left[\frac{i}{4}a^2\right], \tag{27}$$

it follows that the two Hamiltonians H^{os} and H^r are allied to each other as

$$\rho^{-1}H^{os}\rho = i\frac{\hbar\omega}{2}\left(a^{\dagger 2} + a^2\right) = -iH^r. \tag{28}$$

One can verify that in the case of the inverted oscillator, the form of Hamiltonian in the last equation looks like

$$H^r = \frac{i\hbar\omega}{2}(\bar{A}A + A\bar{A}), \tag{29}$$

where the pseudo-ladder operators $\left(A, \bar{A}\right)$ are linked to the ladder operators (6) through the transformation

$$A = \rho^{-1} a \rho = a + i a^\dagger, \tag{30}$$

$$\bar{A} = \rho^{-1} a^\dagger \rho = \frac{1}{2}\left(a^\dagger + i a\right), \tag{31}$$

and satisfy the following commutation relation $\left[A, \bar{A}\right] = 1$.Then, we can deduce that their Fock eigenstates $|n^r\rangle$ are related to $|n^{os}\rangle$ by the invertible operator ρ as

$$|n^r\rangle = \rho^{-1}|n^{os}\rangle. \tag{32}$$

For instance, the pseudo-Hermitian quadratures (X, P) corresponding in the Hermitian system to the coordinate and momentum operators (x, p) (see Eqs. (8)) respectively, are now

$$X = \rho^{-1} x \rho = \sqrt{\frac{\hbar}{2\omega m}}\rho^{-1}\left(a^\dagger + a\right)\rho$$
$$= \sqrt{\frac{\hbar}{2\omega m}}\left(A + \bar{A}\right), \tag{33}$$

$$P = \rho^{-1} p \rho = i\sqrt{\frac{\hbar\omega m}{2}}\rho^{-1}\left(a^\dagger - a\right)\rho$$
$$= i\sqrt{\frac{\hbar\omega m}{2}}\left(\bar{A} - A\right). \tag{34}$$

Knowing that any observable o in the Hermitian system possesses a counterpart O in the pseudo-Hermitian system given by

$$O = \rho^{-1} o \rho, \tag{35}$$

one can deduce the useful representation of $\left(A, \bar{A}\right)$ in terms of (X, P) as

$$A = \sqrt{\frac{m\omega}{2\hbar}}X + i\frac{1}{\sqrt{2m\hbar\omega}}P, \tag{36}$$

$$\bar{A} = \sqrt{\frac{m\omega}{2\hbar}}X - i\frac{1}{\sqrt{2m\hbar\omega}}P. \tag{37}$$

Thereby, the Hamiltonian (29) can be written in terms of X and P as

$$H^r = \frac{i}{2}\left(\frac{P^2}{m} + m\omega^2 X^2\right). \tag{38}$$

This leads to the equations of motion of the inverted oscillator. Indeed, using the Heisenberg equations of motion and $[X, P] = i\hbar$, we have for X and P:

$$\frac{dX}{dt} = \frac{1}{i\hbar}\left[X, \frac{i}{2}\left(\frac{P^2}{m} + m\omega^2 X^2\right)\right] = i\frac{P}{m}.$$
$$\frac{dP}{dt} = \frac{1}{i\hbar}\left[P, \frac{i}{2}\left(\frac{P^2}{m} + m\omega^2 X^2\right)\right] = -im\omega^2 X. \tag{39}$$

Taking another time derivative of dX/dt, we get the usual equation of motion for the inverted oscillator

$$\frac{d^2X}{dt^2} - \omega^2 X = 0, \tag{40}$$

4. Coherent states for the inverted oscillator

The best way to present the inverted coherent states is by translating their definitions into the language of the coherent states of the harmonic oscillator which are summarized in what follows. Coherent states, or semi-classic states, are remarkable quantum states that were originally introduced in 1926 by Schrödinger for the Harmonic oscillator [31] where the mean values of the position and momentum operators in these states have properties close to the classical values of the position $x_c(t)$ and the momentum $p_c(t)$. In particular, the coherent states of the harmonic oscillator$|\alpha^{os}\rangle$ [32]- [34] may be obtained in different but equivalent ways:

(i) as eigenstates of the annihilation operator;

$$a|\alpha\rangle^{os} = \alpha|\alpha\rangle^{os}, \tag{41}$$

with eigenvalues $\alpha \in C$.

(ii) as a displacement of the vacuum $|0\rangle^{os}$, where the displacement operator

$$D^{os}(\alpha) = \exp[\alpha^* a^\dagger - \alpha a], \tag{42}$$

can be used to generate the coherent state

$$|\alpha\rangle^{os} = D^{os}(\alpha)|0\rangle^{os}, \tag{43}$$

(iii) as states that minimize the Heisenberg uncertainty principle

$$\Delta x \Delta p = \frac{\hbar}{2}. \tag{44}$$

Coherent states form an over-complete set of states. The identity operator I is written in terms of coherent states as

$$\frac{1}{\pi}\int |\alpha\rangle^{os\ os}\langle\alpha|\, d^2\alpha = I. \tag{45}$$

The solution for the harmonic oscillator Hamiltonian for an initial coherent state is given in the following simple form

$$|\alpha, t\rangle^{os} = e^{-i\frac{\omega t}{2}}\left|\alpha e^{-i\omega t}\right\rangle^{os}, \tag{46}$$

i.e., a coherent state that rotates with the harmonic oscillator frequency.

In analogy with the usual coherent states, we use the pseudo-annihilation $A = \rho^{-1}a\rho$ and pseudo-creation $\bar{A} = \rho^{-1}a^{\dagger}\rho$ operators which are very convenient to study the inverted coherent states. We emphasize the use of the metric $\eta = \rho^{\dagger}\rho$ operator such as $(iH^{os})^{\dagger} = \eta\,(iH^{os})\,\eta^{-1}$, *i.e.* (iH^{os}) is η-pseudo-Hermitian with respect to a positive-definite inner product defined by $\langle .,.\rangle_{\eta} = \langle .|\eta|.\rangle$:

$$^{r}\langle n|\,\eta\,|m\rangle^{r} = {}^{os}\langle n\,|m\rangle^{os} = \delta_{mn}, \tag{47}$$

which indicates that the Fock states are linked to each other as

$$|n\rangle^{r} = \rho^{-1}\,|n\rangle^{os}, \tag{48}$$

additionally, the vacuum state of the inverted oscillator $|0\rangle^{r}$ ($A\,|0\rangle^{r} = 0$) and the vacuum state of the harmonic oscillator $|0\rangle^{os}$ are related as $|0\rangle^{r} = \rho^{-1}\,|0\rangle^{os}$.

The coherent states for the inverted harmonic oscillator are defined as eigenstates of the corresponding pseudo-annihilation operator A

$$A\,|\alpha\rangle^{r} = \alpha\,|\alpha\rangle^{r}, \quad \alpha\in\mathbb{C}. \tag{49}$$

with

$$|\alpha\rangle^{r} = \rho^{-1}\,|\alpha\rangle^{os}. \tag{50}$$

Particularly, the normalization condition

$$^{os}\langle\alpha\,|\alpha\rangle^{os} = 1, \tag{51}$$

leads to

$$^{r}\langle\alpha|\,\eta\,|\alpha\rangle^{r} = 1, \tag{52}$$

and then the integral

$$\frac{1}{\pi}\int_{\mathbb{C}}\rho\,|\alpha\rangle^{r}\;{}^{r}\langle\alpha|\,\rho^{+}d\alpha^{*}d\alpha = I, \tag{53}$$

is an identity operator.

These inverted coherent states $|\alpha\rangle^{r}$ can also be generated respectively from the vacuum states $|0\rangle^{r}$ by the action of pseudo-displacement operator $D^{r}(\alpha)$,

$$|\alpha\rangle^{r} = D^{r}(\alpha)\,|0\rangle^{r} = \exp\left[\alpha\bar{A} - \alpha^{*}A\right]|0\rangle^{r}, \tag{54}$$

we note that $D^{r}(\alpha)$ is related to $D^{os}(\alpha)$ as

$$D^{r}(\alpha) = \rho^{-1}D^{os}(\alpha)\,\rho. \tag{55}$$

Using the Hamiltonian (29), we deduce the evolution of an initial inverted coherent state in the following simple form

$$\begin{aligned}|\alpha,t\rangle^{r} &= e^{-i/\hbar H^{r}t}\,|\alpha\rangle^{r}\\ &= e^{-|\alpha|^{2}/2}e^{\omega t/2}e^{\omega\bar{A}At}\sum_{n}\frac{(\alpha)^{n}}{\sqrt{n!}}\,|n\rangle^{r}.\end{aligned} \tag{56}$$

Introducing $e^{\omega\bar{A}At}$ into the sum, and using the fact that the states $|n\rangle^{r}$ are eigenstates of the number operator$\bar{A}A$, we have

$$\begin{aligned}|\alpha,t\rangle^{r} &= e^{-\left|\alpha e^{\omega t}\right|^{2}/2}\sum_{n}\frac{(\alpha e^{\omega t})^{n}}{\sqrt{n!}}\,|n\rangle^{r}\\ &= e^{\omega t/2}\left|\alpha e^{\omega t}\right\rangle^{r}.\end{aligned} \tag{57}$$

Since our aim is to compute the Heisenberg uncertainty relations in the position and the momentum, it is required to calculate the expectation values of the canonical variables and their squares in the inverted coherent states. Then, by using the relation (35) in the non-Hermitian system, the expectation value of an arbitrary operator $O = X, X^{2}, P$ and P^{2} can be evaluated from

$$\langle O\rangle_{\eta} = {}^{r}\langle\alpha,t|\,\eta O\,|\alpha,t\rangle^{r} = {}^{r}\langle\alpha,t|\,\rho^{+}o\rho\,|\alpha,t\rangle^{r} = e^{-\frac{|\alpha e^{\omega t}|^{2}}{2}}\sum_{n}\sum_{m}\frac{(\alpha^{*}e^{\omega t})^{m}}{\sqrt{m!}}\frac{(\alpha e^{\omega t})^{n}}{\sqrt{n!}}\;{}^{os}\langle m|\,o\,|n\rangle^{os}. \tag{58}$$

Using the above equation, the expectation values of X and P in the state $|\alpha,t\rangle^{r}$ are easily evaluated:

$$\langle X\rangle_{\eta} = e^{-\left|\alpha e^{\omega t}\right|^{2}}\sum_{n}\sum_{m}\frac{(\alpha^{*}e^{\omega t})^{m}}{\sqrt{m!}}\frac{(\alpha e^{\omega t})^{n}}{\sqrt{n!}}\sqrt{\frac{\hbar}{2\omega m}}{}^{os}\langle m|\left(a^{\dagger}+a\right)|n\rangle^{os} = \sqrt{\frac{\hbar}{2m\omega}}\left[\alpha+\alpha^{*}\right]e^{\omega t}, \tag{59}$$

$$\langle P\rangle_{\eta} = e^{-\left|\alpha e^{\omega t}\right|^{2}}\sum_{n}\sum_{m}\frac{(\alpha^{*}e^{\omega t})^{m}}{\sqrt{m!}}\frac{(\alpha e^{\omega t})^{n}}{\sqrt{n!}}i\sqrt{\frac{\hbar\omega m}{2}}\;{}^{os}\langle m|\left(a^{\dagger}-a\right)|n\rangle^{os} = -i\sqrt{\frac{m\omega\hbar}{2}}\left[\alpha-\alpha^{*}\right]e^{\omega t}, \tag{60}$$

and follow classical physics; *i.e.*

$$\langle X\rangle_{\eta} = x_{c}, \quad \langle P\rangle_{\eta} = p_{c}, \tag{61}$$

where the subscript c indicate classical. This is why we call these inverted coherent states ”quasi-classical states”.

Let us now evaluate the uncertainty in the position and the momentum.

$$\left\langle X^2\right\rangle_\eta = \langle\alpha,t|\,\eta X^2\,|\alpha,t\rangle^r = \frac{\hbar}{2m\omega}\left[\alpha^2 e^{2\omega t}+\alpha^{*2}e^{2\omega t}+2\left(|\alpha|^2 e^{2\omega t}+\frac{1}{2}\right)\right], \tag{62}$$

$$\left\langle P^2\right\rangle_\eta = \langle\alpha,t|\,\eta P^2\,|\alpha,t\rangle^r = \frac{-im\omega\hbar}{2}\left[\alpha^2 e^{2\omega t}+\alpha^{*2}e^{2\omega t}-2\left(|\alpha|^2 e^{2\omega t}+\frac{1}{2}\right)\right]. \tag{63}$$

It is well known that the position uncertainty can be derived from $\Delta X = \sqrt{\langle X^2\rangle_\eta - \langle X\rangle_\eta^2}$. Then using (59) and (62), we have

$$\Delta X = \sqrt{\frac{\hbar}{2m\omega}}.$$

Similarly, from Eqs. (60) and (63), we also have the momentum uncertainty such that

$$\Delta P = \sqrt{\frac{m\omega\hbar}{2}}.$$

Thus, the uncertainty product for canonical variables X and P is given by

$$\Delta X\Delta P = \frac{\hbar}{2}.$$

Therefore, the inverted coherent states are a minimum-uncertainty states and the time evolution of an initially inverted coherent state can be regarded as the quantum analog of a classical trajectory.

5. Conclusion

We have briefly summarized in Sec. 2, some properties of the standard harmonic and inverted oscillators.

We have proposed a scheme that permits relating a regular harmonic oscillator to an inverted oscillator by using a time-independent Dyson metric which allowed us to introduce the pseudo-annihilation $A = \rho^{-1}a\rho$ and pseudo-creation $\bar{A} = \rho^{-1}a^\dagger\rho$ operators associated to the inverted harmonic oscillator. These operators are the basis of the definition of coherent states for inverted oscillator and their corresponding eigenstates and eigenvalues. Once the Dyson operator has been introduced, and therefore the metric operators, it is straightforward to extend these considerations to the associated eigenstates and inner product structures on the physical Hilbert space. Some of the findings are treated by the Gaussian wave packet (in the x-representation) associated to the generalized coherent state in Ref. [35].

Coherent states of the inverted harmonic oscillator are constructed in different forms:

(1) as eigenstates of the pseudo-annihilation operator A;

(2) as a pseudo-displacement of the inverted vacuum $\exp\left[\alpha\overline{A}-\alpha^* A\right]|0\rangle^r$,

(3) as states whose averages follow the classical trajectories of X, P and H^r.

However, the coherent states for the inverted oscillator constitute ”minimum uncertainty” wave packets. Therefore, the time evolution of an initially coherent state can be regarded as the quantum analog of a classical trajectory.

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i. It is useful to note that the following simplified transformations (27) has been introduced in Ref. [17] as footnote.

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