Classical and quantum dynamics of over-damped non linear systems

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Overdamping is a regime in which friction is sufficiently large that the motion either decays to its equilibrium position or it crosses the equilibrium position exactly once before returning monotonically towards the equilibrium position. The phenomena of overdamping has been studied classically and quantum mechanically only for the case of the linear damped harmonic oscillator. Here we study the classical and quantum dynamics of a family of over-damped non linear systems. The main objective of this paper is to find a Lagrangian and Hamiltonian framework to study over-damped non linear systems and to show that a quantum mechanical description can be developed in the momentum representation. Our results reduce to the well known solution of the linear damped harmonic oscillator when the non linear part is set to zero.

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1. Introduction

It is very well known that the linear damped harmonic oscillator can be characterized from the physical point of view in three different cases: underdamped, over-damped and critically damped [1]. The important equation that describes these type of motions is given by

$$\ddot{x}+2\gamma\dot{x}+\omega^2x=0, \tag{1}$$

where γ and ω are the damping and angular frequency constant coefficients, respectively. When the damping coefficient is zero the motion is purely oscillatory with frequency ω. The oscillations can be suppressed by adding a small damping to the system in such a way that the oscillations have a very slow decay to its equilibrium position and can cross zero infinitely often before settling to zero as $t\to\infty$, this type of motion is known as underdamped. For the case when the damping is sufficiently large the motion either decays to its equilibrium position or it crosses the equilibrium position exactly once before returning monotonically towards the equilibrium position as $t\to\infty$, this type of motion is known as over-damped. Critically damped motion represents the boundary between oscillatory and non-oscillatory behavior of the system [2].

In a remarkable paper Chandrasekar *et al*. developed a time-independent Lagrangian that yield the correct equations of motion for a linear damped harmonic oscillator having one degree of freedom [3]. This result was obtained by using a modified Prelle-Singer method by obtaining first time-independent integrals of motion for the three different cases of the linear damped harmonic oscillator [4,5]. Using these constants of motion, different forms of the Lagrangian and Hamiltonian were given depending on whether the system was overdamped, underdamped, or critically damped and the resultant canonical equations are shown to lead to the standard dynamical description. In particular, for the over-damped case the Hamiltonian which describes the equation of motion (1) takes the following peculiar form [6]

$$H(x,p)=\frac{\omega_1}{\omega_1-\omega_2}p^{1-\omega_2/\omega_1}-i\omega_1xp, \tag{2}$$

where $\omega_{1,2}$ are the eigenfrequencies which are obtained by substituting $x(t)=e^{-i\omega_{1,2}t}$ into Eq. (1) in order to get

$$\omega_{1,2}=-i\gamma\pm\sqrt{\omega^2-\gamma^2}. \tag{3}$$

In order to have overdamping and a real valued Hamiltonian we need to have purely imaginary values for $\omega_{1,2}$, therefore we must choose $\gamma^2>\omega^2$. Using the Legendre transformation one con obtain the Lagrangian associated with the Hamiltonian given in Eq. (2), therefore we have

$$\mathcal{L}(x,\dot{x})=\frac{\omega_2}{\omega_2-\omega_1}(\dot{x}+i\omega_1x)^{1-\omega_1/\omega_2}. \tag{4}$$

Note that if $\gamma=0$ then $\omega=\omega_1=-\omega_2$ which gives us in return the standard Lagrangian for the simple harmonic oscillator plus a gauge factor which do not affects the equation of motion. With a given Lagrangian we can then obtain the equations of motion of the system by using the Euler-Lagrange equations [7]

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right)-\frac{\partial\mathcal{L}}{\partial x}=0. \tag{5}$$

Substituting the Lagrangian given in Eq. (4) into the Euler-Lagrange equations we get the following equation of motion

$$\ddot{x}+i(\omega_1+\omega_2)\dot{x}-\omega_1\omega_2x=0, \tag{6}$$

and using the fact that

$$\omega_1+\omega_2=-2i\gamma,\quad \text{and}\quad \omega_1\omega_2=-\omega^2, \tag{7}$$

then the equation of motion given by the Euler-Lagrange equation reduces to the Eq. (1).

In this paper we show in Sec. 2 that the phenomena of overdamping can be studied classically for the following family of non linear systems

$$\ddot{x}+i(\omega_1+\omega_2)\dot{x}-\omega_1\omega_2 x+i\Gamma\omega_2(\dot{x}+i\omega_1 x)^{1+\omega_1/\omega_2}=0, \quad (8)$$

where Γ is a real constant introduced to ensure dimensional consistency. In particular we analytically solved Hamilton's equations in order to obtain the solution of the non linear system given in Eq. (8) and show that the solutions are asymptotically stable to the origin and which are zero at most once for $0<t<\infty$. In Sec. 3 we approach the problem of quantization of the over-damped non linear system in a rigged Hilbert space by finding a solution to the time-independent Schrödinger equation in the momentum representation for Eq. (8). The conclusions are summarized in the last section.

2. Hamiltonian for an over-damped non linear system

Many dynamical systems in physics in general are described by non linear second order differential equations [8]. In this section we are going to study the effect of an additional non linear term to the linear damped harmonic oscillator. Let us now consider the following non linear autonomous system given by

$$\ddot{x}+i(\omega_1+\omega_2)\dot{x}-\omega_1\omega_2 x+i\Gamma\omega_2(\dot{x}+i\omega_1 x)^{1+\omega_1/\omega_2}=0. \quad (9)$$

It is easy to convince oneself, by using the Euler-Lagrange equations, that a Lagrangian for Eq. (9) is given by

$$\mathcal{L}(x,\dot{x})=\frac{\omega_2}{\omega_2-\omega_1}(\dot{x}+i\omega_1 x)^{1-\omega_1/\omega_2}+\Gamma\left(\dot{x}+i\omega_1 x\right). \quad (10)$$

Using the Lagrangian given in Eq. (10) we can obtain the canonical momentum which is given by

$$p=(\dot{x}+i\omega_1 x)^{-\omega_1/\omega_2}+\Gamma, \quad (11)$$

and by using the Legendre transformation one can then obtain the Hamiltonian which is given by

$$H(x,p)=\frac{\omega_1}{\omega_1-\omega_2}(p-\Gamma)^{1-\omega_2/\omega_1}-i\omega_1 xp. \quad (12)$$

Interestingly, the Hamiltonian obtained for the non linear system given by Eq. (9) has almost the same form as the one obtained for the linear system given by Eq. (1).

For the Hamiltonian given in Eq. (12) we can write down Hamilton's equations which are given by

$$\dot{x}=\frac{\partial H}{\partial p}=(p-\Gamma)^{-\omega_2/\omega_1}-i\omega_1 x, \quad (13)$$

$$\dot{p}=-\frac{\partial H}{\partial x}=i\omega_1 p. \quad (14)$$

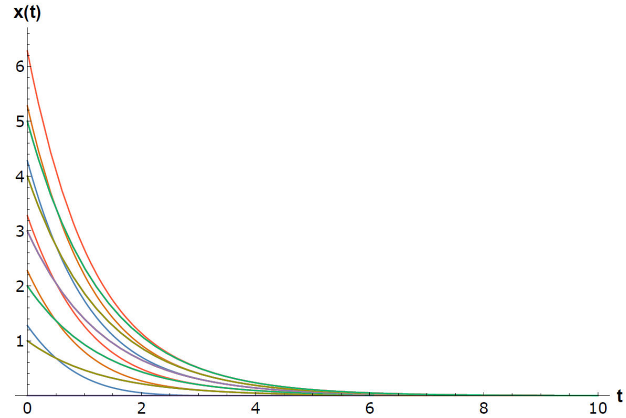

FIGURE 1. The graph shows the position as a function of time for the non linear over-damped equation given in Eq. (9). We have set the damping coefficient $\gamma=3$, the angular frequency $\omega=2$ and $\Gamma=-1$. The plot shows the solution given in Eq. (15) for several values of the initial parameters x_0 and p_0, respectively.

By solving for $p(t)$ in Eq. (14) and substituting into Eq. (13) we obtain a non homogeneous linear differential equation with the following solution

$$x(t)=x_0 e^{-i\omega_1 t}+\frac{ie^{-i\omega_1 t}}{p_0(\omega_2-\omega_1)} \times\left(p_0 e^{i\omega_1 t}-\Gamma\right)^{1-\omega_2/\omega_1}. \quad (15)$$

In Fig. 1 we plot the solution given in Eq. (15) for a given value of the damping and frequency constants and several values of the initial conditions x_0 and p_0. Note how all solutions approach asymptotically to the origin as $t\to\infty$.

3. Quantization in a rigged Hilbert space

Having obtained the Hamiltonian of the non linear autonomous system, we now proceed to see if it is possible to solve the Schrödinger equation [9,10]. For this purpose, we first rewrite the classical Hamiltonian given in Eq. (12) in a symmetrized form so as to ensure hermiticity, *i.e.* [11]

$$H(x,p)=\frac{\omega_1}{\omega_1-\omega_2}(p-\Gamma)^{1-\omega_2/\omega_1} -\frac{i\omega_1}{2}(xp+px). \quad (16)$$

Using the momentum representation, where $x=i\hbar\partial_p$ is an operator and p is a c number, and looking for stationary solutions of the following form $\Psi(p,t)=\psi(p)e^{-iEt/\hbar}$ in the time dependent Schrödinger equation [12]

$$i\hbar\frac{\partial\Psi(p,t)}{\partial t}=H\Psi(p,t), \quad (17)$$

we obtain the time independent Schrödinger equation

$$E\psi(p)=H\psi(p). \quad (18)$$

Using the commutation relation $[x,p] = i\hbar$ we can write down the time independent Schrödinger equation in the following form

$$\left[\frac{\omega_1}{\omega_1-\omega_2}(p-\Gamma)^{1-\omega_2/\omega_1} + \frac{\hbar\omega_1}{2}\left(1+2p\frac{d}{dp}\right)\right]\psi(p) = E\psi(p). \tag{19}$$

By using the following transformation

$$\psi(p) = \tilde{\psi}(p)\exp\left[\frac{1}{\hbar(\omega_2-\omega_1)} \times \int \frac{1}{p}(p-\Gamma)^{1-\omega_2/\omega_1}\, dp\right], \tag{20}$$

into the Schrödigner equation given in (19) we get the following differential equation

$$p\frac{d\tilde{\psi}}{dp} = \left(\frac{E}{\hbar\omega_1}-\frac{1}{2}\right)\tilde{\psi}, \quad -\infty < E < \infty. \tag{21}$$

The above eigenvalue equation has the same form as that of the eigenvalue equation in the coordinate space for the toy model for quantum damping studied by Chruscinski in a rigged Hilbert space [13]. Consequently the generalized eigenfunctions can be given as

$$\tilde{\psi}(p) = \sqrt{\frac{1}{2\pi\hbar|\omega_1|}}\, p_{\pm}^{\left(\frac{E}{\hbar\omega_1}-\frac{1}{2}\right)}, \tag{22}$$

where following Ref. [13] the tempered-distributions are defined as

$$p_{+}^{\lambda} = \begin{cases} p^{\lambda}, & \text{if } p \geq 0 \\ 0, & \text{if } p < 0 \end{cases}, \tag{23}$$

$$p_{-}^{\lambda} = \begin{cases} 0, & \text{if } p \geq 0 \\ |p|^{\lambda}, & \text{if } p < 0 \end{cases}. \tag{24}$$

Substituting Eq. (22) into Eq. (20) we have the general solution for our problem which is given by

$$\psi(p) = \sqrt{\frac{1}{2\pi\hbar|\omega_1|}}\, p_{\pm}^{\left(\frac{E}{\hbar\omega_1}-\frac{1}{2}\right)} \times \exp\left[\frac{1}{\hbar(\omega_2-\omega_1)}\int \frac{1}{p}(p-\Gamma)^{1-\omega_2/\omega_1}\, dp\right]. \tag{25}$$

Note that Eq. (25) reduces to the solution given by Chandrasekar *et al.* for the linearly over-damped harmonic oscillator case when $\Gamma \to 0$ [3].

Chruscinski has shown that the complex eigenvalues associated to the momentum wave function given in Eq. (25) correspond to the poles of energy eigenvectors when continued to the complex eigenvalue plane and that the wave function may be interpreted as resonant states which are responsible for the irreversible quantum dynamics. Detailed description may be found in Ref. [13].

4. Conclusions

In this article we have shown a Lagrangian and Hamiltonian framework valid for the study of the classical and quantum regime of an over-damped autonomous non linear system. Furthermore, our analytical solutions reduce to the ones obtained by Chandrasekar *et al.* for the linear damped harmonic oscillator when the non linear part is set to zero. We expect that further investigation at the classical and quantum level of over-damped non linear systems can reveal important features of these systems.

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