Radial solution of Schrödinger equation with Hulthen-Yukawa-Inverse quadratic potential in a Point-Like defect under AB-flux field

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In this paper, we determine the approximate eigenvalue solution of the non-relativistic wave equation in the presence of the Aharonov-Bohm flux field with Hulthen-Yukawa-Inverse Quadratic potential in a topological defect via point-like global monopole (PGM) geometry. We use the Greene-Aldrich improved approximation scheme into the centrifugal and reciprocal terms appear in the radial Schrödinger equation. We then solve this radial equation using the parametric Nikiforov-Uvarov method and analyze the effects on the eigenvalue solution. We see that the energy levels and the radial wave functions get modified by the topological defect of a point-like global monopole and the magnetic flux field that shows an analogue of the Aharonov-Bohm effect for the bound state. Finally, we utilize the eigenvalue solution to some potential models, such as Hulthen potential, Hulthen plus Yukawa potential, and Hulthen plus inverse quadratic potential and discuss the results.

Keywords: Topological defect; non-relativistic wave equation; solutions of wave equations: bound state; geometric quantum phase; special function; interaction potential.

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1. Introduction

The exact and approximate eigenvalue solutions of the non-relativistic Schrödinger equation (SE) with different potential models are significant in physics and chemistry because they contain all the necessary information of a quantum system. In the literature, H-atom and harmonic oscillator problems discussed in many textbooks are two of several exactly solvable quantum mechanical problems in classical and quantum physics [1–4]. Several authors have obtained the exact and approximate solutions to the radial Schrödinger equation by considering various kinds of potential of physical interest, such as Yukawa potential [5–8], Hulthen potential [9–12], Hartmann potential [13–15], Manning-Rosen (MR) potential [16, 17], Hylleraas potential [18, 19], Eckart potential [20, 21], Morse potential [22, 23], Rosen-Morse potential [24–26], Killingbeck potential [27], Kratzer potential [28–32], Eckart-Hellmann potential [34] and many more. Noted that all the above investigations were done in the flat space background both in the cylindrical and spherical systems.

The exact and approximate eigenvalue solutions have been found using several methods or techniques in the literature. Among them, the parametric Nikiforov-Uvarov method [35] is one of the successful methods applied to obtain the energy levels and the wave functions of a second-order homogeneous differential equation (see, Refs. [6–8, 11, 12, 32, 33, 36–40]). According to this method, the wave functions of a second-order differential equation of the following form [35]

$$\psi''(s) + \frac{(c_1 - c_2\, s)}{s\,(1 - c_3\, s)}\,\psi'(s) + \frac{(-\xi_1\, s^2 + \xi_2\, s - \xi_3)}{s^2\,(1 - c_3\, s)^2}\,\psi(s) = 0 \tag{1}$$

are given by

$$\psi(s) = s^{c_{12}}(1 - c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}} \times P_n^{\left(c_{10}-1,\, \frac{c_{11}}{c_3} - c_{10} - 1\right)}(1 - 2\, c_3 s). \tag{2}$$

And the energy eigenvalue equation is

$$c_2\, n - (2\, n + 1)\, c_5 + (2\, n + 1)\,(\sqrt{c_9} + c_3\,\sqrt{c_8}) + n\,(n - 1)\, c_3 + c_7 + 2\, c_3\, c_8 + 2\,\sqrt{c_8\, c_9} = 0. \tag{3}$$

The parameters $c_4, \ldots, c_{13}$ are obtained from the six parameters $c_1, \ldots, c_3$ and $\xi_1, \ldots, \xi_3$ as follows:

$$\begin{aligned}
&c_4 = \frac{1}{2}\,(1 - c_1), \quad c_5 = \frac{1}{2}\,(c_2 - 2\, c_3), \quad c_6 = c_5^2 + \xi_1,\\
&c_7 = 2\, c_4\, c_5 - \xi_2, \qquad c_8 = c_4^2 + \xi_3,\\
&c_9 = c_6 + c_3\, c_7 + c_3^2\, c_8, \qquad c_{10} = c_1 + 2\, c_4 + 2\,\sqrt{c_8},\\
&c_{11} = c_2 - 2\, c_5 + 2\,(\sqrt{c_9} + c_3\,\sqrt{c_8}),\\
&c_{12} = c_4 + \sqrt{c_8}, \qquad c_{13} = c_5 - (\sqrt{c_9} + c_3\,\sqrt{c_8}).
\end{aligned} \tag{4}$$

The study of topological defects via a point-like global monopole in quantum system has been done only in a handful of works in the literature. In the relativistic limit, these studies are the hydrogen and pionic atoms [41], quantum motions of a spin-zero particle with potential under the AB-flux field [42], solution of the Dirac equation [43], the exact solution of the DKP equation under the AB-flux field and Coulomb potential [44], and the Klein-Gordon oscillator and its generalization Refs. [36, 37, 45, 46]. On the other hand, studies of the non-relativistic Schrödinger equation [47], quantum scattering of charged particles [48], harmonic oscillator problem [49], a harmonic oscillator with Cornell-type potential [50], a harmonic oscillator with Mie-type potential in the presence of the AB-flux field [51], non-relativistic equation with Kratzer and Morse potential [52], quantum motions of particles with generalized Morse potential [53], non-relativistic particles interact with diatomic molecular potential in the presence of the AB-flux field [54], quantum dynamics of non-relativistic particles with pseudoharmonic- and Mie-type potentials in the presence of the AB-flux field [55], and non-relativistic particles interact with generalized q-deformed Hulthen plus Coulomb and inverse quadratic Yukawa potential [56] are known in the non-relativistic limit. The presence of topological defect modifies the eigenvalue solutions in comparison with the flat space results and broke the degeneracy. Thus, the physical properties of the quantum system are changed by the global features of point-like global monopole geometry. In addition, the presence of external magnetic and the quantum flux fields also shift the eigenvalue solutions if one would considered in a quantum system in addition to the topological defects. The dependence of the eigenvalue solutions on the geometric quantum phase shows an analogue to the Aharonov-Bohm effect [57,58] for the bound state. This AB-effect is a quantum mechanical phenomenon where the particles confined by the AB-flux field experience zero electric and magnetic fields everywhere except at the origin. In this analysis, we study the non-relativistic Schrödinger equation in three dimensions in the presence of the AB-flux field with potential in a point-like global monopole. The considered potential is the superposition of Hulthen potential (HP), Yukawa potential (YP), and Inverse Quadratic potential (IQP) which have several applications in different branches of physics and chemistry. As the chosen potential is exponential-type, such as Hulthen potential and Yukawa potential, one should employ a suitable approximation scheme into the centrifugal and reciprocal terms appear in the radial equation in order to obtain its solution. In this analysis, among many approximation schemes we use one such scheme called the Greene-Aldrich improved approximation scheme [59] that gives us a good approximation to $\sim 1/r$ and $\sim 1/r^2$ for small values of the screening parameter $\delta \ll 1$. The small values of the parameter $\delta \ll 1$ correspond to a short-range potential. We then solve this radial equation using the parametric NU-method and obtain the approximate eigenvalue solution. We see that the eigenvalue solution gets influenced by the topological defect of a point-like global monopole and the magnetic flux field with potential. The presence of a topological defect shifts the energy levels and these get modified in comparison with the flat space result with this chosen potential.

A static and spherically symmetric space-time describing a point-like global monopole geometry in the spherical coordinates (r, θ, ϕ) in three dimensions is given by [41,42,46,49–54]

$$ds^2_{3D} = \frac{dr^2}{\alpha^2} + r^2\,(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{5}$$

where $\alpha < 1$ represents the topological defect parameter of a point-like global monopole (PGM). For $\alpha \to 1$, this space-time geometry reduces to Minkowski flat space.

This paper is organised as follows: in Sec. 2, we will discuss 3D radial Schrödinger non-relativistic wave equation in the presence of the AB-flux field with potential in a conical singularity space-time. Then, we solve the radial equation through a suitable approximation scheme using the parametric NU-method and obtain the eigenvalue solution; in Sec. 3, we utilize the eigenvalue solution to some individual as well as combined potential models; in Sec. 4, we present our results. We have used the system of units where $c = 1 = \hbar = G$.

2. Eigenvalue Solution of Non-Relativistic Equation Under AB-flux Field with Potential in a Point-like Defect

In this section, we study the quantum motions of non-relativistic particles confined by the AB-flux field with potential in a topological defect geometry produced by point-like global monopole (PGM). We solve the three dimensional radial Schrödinger equation and discuss the effects of topological defects and the magnetic flux with potential on the eigenvalue solution.

We begin this paragraph with time-independent Schrödinger wave equation with a spatial-dependent interaction potential $V(r)$ described by the following wave equation [42,49–55]

$$\left[-\frac{1}{2M}\left(\frac{1}{\sqrt{g}}\,D_i\left(\sqrt{g}\,g^{ij}\,D_j\right)\right) + V(r)\right]\Psi = E\,\Psi, \tag{6}$$

where M is the mass of the non-relativistic particles, $D_i \equiv (\partial_i - i\,e\,A_i)$ [1,2], $i = 1, 2, 3$ where e is the electric charge, A_i is the electromagnetic three-vector potential, g_{ij} is the metric tensor with g^{ij} its inverse. For the space-time (5), its determinant will be $g = |g_{ij}| = r^4\,\sin^2\theta/\alpha^2$.

In this analysis, we have chosen the following electromagnetic three-vector potential $\vec{A}$ given by Refs. [42,44,46,51,54,55]

$$A_r = 0 = A_\theta, \quad A_\phi = \frac{\Phi_{AB}}{2\,\pi\,r\,\sin\theta}, \tag{7}$$

where $\Phi_{AB} = const = \Phi\,\Phi_0$ is the Aharonov-Bohm magnetic flux field, $\Phi_0 = 2\,\pi\,e^{-1}$ is the quantum of magnetic flux, and Φ is the amount of magnetic flux which is a positive integer. Several authors have studied the quantum mechanical problems in the presence of the AB-flux field in the literature (see, Ref. [42,44,46] and related references therein). It is well-known that the dependence of the eigenvalue solution on the geometric phase shows an analogue of the Aharonov-Bohm effect [57,58] for the bound state.

In the literature, it is well-known that the total wave function Ψ can always express in terms of different variables by the method of separation of variables. Since we are dealing with the spherical system, a possible total wave function

$\Psi(t,r,\theta,\phi)$ in terms of a radial wave function $\psi(r)$ can be expressed as

$$\Psi(t,r,\theta,\phi)=e^{-i\,E\,t}\,Y_{l,m}(\theta,\phi)\,\frac{\psi(r)}{r}, \tag{8}$$

where E is the energy of the particle, $Y_{l,m}(\theta,\phi) = A_{l,m}(\theta)\,B_m(\phi)$ is the spherical harmonic functions, and l,m are respectively the orbital and magnetic quantum numbers.

Thereby, substituting Eqs. (7)–(8) into the Eq. (6) and expressing in the space-time background (5), we have obtained the following wave equation in terms of the radial function $\psi(r)$ as:

$$\psi''(r)+\frac{1}{\alpha^2}\left[2\,M\,(E-V(r))-\frac{l'\,(l'+1)}{r^2}\right]\psi(r)=0,$$
$$l'=(l-\Phi), \tag{9}$$

where we have used various eigenvalues of operators given in Refs. [51, 54–56].

In this analysis, we have considered the superposition of Hulthen potential [9–12], Yukawa potential [5–8], and inverse quadratic potential [2, 60] given by

$$V_{HYIQ}(r)=V_H(r)+V_Y+V_{IQ},\quad V_H=-\frac{V_0\,e^{-2\,\delta\,r}}{(1-e^{-2\,\delta\,r})},$$
$$V_Y=-\frac{a}{r}\,e^{-\delta\,r},\qquad V_{IQ}=\frac{b}{r^2}, \tag{10}$$

where V_0 is the potential depth, a,b are the potential strength parameters, and δ is the screening parameter. It is worth mentioning that a common screening parameter is chosen such that one can employ a suitable approximation scheme into the centrifugal term and would be able to solve the radial equation. The Hulthen potential [9, 10] is a short-range potential which behaves like a Coulomb potential for small values of r and decreases exponentially for large values of r. This Hulthen potential has widely been used in many branches of physics and chemistry, such as atomic and molecular physics [61, 62], solid state physics [63], and chemical physics [64]. On the other hand, Yukawa potential [5] which is known as a screened Coulomb potential has great importance in different branches of physics and chemistry, such as in plasma physics, particle and nuclear physics, chemical physics, solid-state physics and atomic physics [15, 65].

Using potential (10), one will have the effective potential of the quantum system given by

$$V_{\text{eff}}(r)=\left[\frac{(l-\Phi)\,(l-\Phi+1)}{2\,M\,\alpha^2\,r^2}\right.$$
$$\left.-\frac{V_0\,e^{-2\,\delta\,r}}{\alpha^2\left(1-e^{-2\,\delta\,r}\right)}-\frac{a}{\alpha^2\,r}\,e^{-\delta\,r}+\frac{b}{\alpha^2\,r^2}\right]. \tag{11}$$

One can see that the effective potential is influenced by the topological defects of the geometry characterised by the parameter α and the magnetic flux field Φ_{AB}.

Thereby, substituting potential (10) into the radial equation (9), we have

$$\psi''(r)+\left[\frac{2\,M\,E}{\alpha^2}+\frac{2\,M\,V_0}{\alpha^2}\,\frac{e^{-2\,\delta\,r}}{(1-e^{-2\,\delta\,r})}+\frac{2\,M\,a}{\alpha^2\,r}\,e^{-\delta\,r}\right.$$
$$\left.-\frac{(l'\,(l'+1)+2\,M\,b)}{\alpha^2}\,\frac{1}{r^2}\right]\psi(r)=0. \tag{12}$$

The radial part of the Schródinger equation for the superposed potential (10) cannot be solved for $l\neq 0$. To obtain the eigenvalue solution of the above equation (12) for $l\neq 0$, we employ a suitable approximation scheme into the centrifugal ($\sim 1/r^2$) as well as the reciprocal terms ($\sim 1/r$) appearing in the radial equation. As one can see, in the superposed potential there is a $1/(1-e^{-2\,\delta\,r})$ term. As a result, while for small values of $\delta \ll$ that correspond to a short-range potential, we have employed an improved approximation scheme given by [59, 66–73]

$$\frac{1}{r}\approx\frac{2\,\delta\,e^{-\delta\,r}}{(1-e^{-2\,\delta\,r})},\quad \frac{1}{r^2}\approx\frac{4\,\delta^2\,e^{-2\,\delta\,r}}{(1-e^{-2\,\delta\,r})^2}. \tag{13}$$

Note that Eq. (13) is not a good approximation to the centrifugal barrier when the screening parameter δ becomes large.

Therefore, using the above improved approximation scheme into the Eq. (12), we have

$$\psi''(r)+\left[\beta-\beta_1\,\frac{e^{-2\,\delta\,r}}{(1-e^{-2\,\delta\,r})^2}\right.$$
$$\left.+\beta_2\,\frac{e^{-2\,\delta\,r}}{(1-e^{-2\,\delta\,r})}\right]\psi(r)=0, \tag{14}$$

where we have set the parameters

$$\beta=\frac{2\,M\,E}{\alpha^2},\quad \beta_1=\frac{4\,\delta^2}{\alpha^2}\left[l'\,(l'+1)+2\,M\,b\right],$$
$$\beta_2=\frac{2\,M}{\alpha^2}\,(V_0+2\,a\,\delta). \tag{15}$$

Let us perform a change of variable via $s=e^{-2\,\delta\,r}$ into the above Eq. (14). We have obtained the following second-order differential equation:

$$\psi''(s)+\frac{(c_1-\,c_2\,s)}{s\,(1-c_3\,s)}\,\psi'(s)+\frac{\Big(-\xi_1\,s^2+\xi_2\,s-\xi_3\Big)}{s^2\,(1-c_3\,s)^2}\,\psi(s)=0. \tag{16}$$

where $c_1=1=c_2=c_3$ and

$$\xi_1=\frac{1}{4\,\delta^2}\,(\beta_2-\beta),\qquad \xi_2=\frac{1}{4\,\delta^2}\,(\beta_2-\beta_1-2\,\beta),\qquad \xi_3=-\frac{\beta}{4\,\delta^2}. \tag{17}$$

Equation (16) is a linear homogeneous second-order differential equation that can be solved using a well-known method called the parametric Nikiforov-Uvarov method discussed earlier. Thereby, comparing Eq. (16) with the Eq. (1), the different parameters are as follows:

$$\begin{aligned}
&c_4=0,\qquad c_5=-\frac{1}{2},\qquad c_6=\frac{1}{4}+\xi_1,\qquad c_7=-\xi_2,\qquad c_8=\xi_3,\qquad c_9=\frac{1}{4}+\xi_1-\xi_2+\xi_3,\\
&c_{10}=1+2\,\sqrt{\xi_3},\qquad c_{11}=2\left(1+\sqrt{\frac{1}{4}+\xi_1-\xi_2+\xi_3}+\sqrt{\xi_3}\right),\qquad c_{12}=\sqrt{\xi_3},\\
&c_{13}=-\frac{1}{2}-\left(\sqrt{\frac{1}{4}+\xi_1-\xi_2+\xi_3}+\sqrt{\xi_3}\right).
\end{aligned} \tag{18}$$

Substituting Eq. (18) into the Eq. (3) and using Eq. (17), one can obtain the following expression of the approximate energy eigenvalue

$$E_{n,l}=-\frac{\alpha^2\,\delta^2}{2\,M}\left[\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(l-\Phi)\,(l-\Phi+1)+2\,M\,b}{\alpha^2}}\right)-\frac{\frac{M}{2\,\alpha^2\,\delta^2}\,(V_0+2\,a\,\delta)}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(l-\Phi)\,(l-\Phi+1)+2\,M\,b}{\alpha^2}}\right)}\right]^2. \tag{19}$$

The radial wave functions are given by

$$\psi_{n,l}(s)=s^{\lambda}\,(1-s)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(l-\Phi)\,(l-\Phi+1)+2\,M\,b}{\alpha^2}}}\,P_n^{\left(2\,\lambda,2\,\sqrt{\frac{1}{4}+\frac{(l-\Phi)\,(l-\Phi+1)+2\,M\,b}{\alpha^2}}\right)}(1-2\,s), \tag{20}$$

where $P_n^{(c,d)}$ are the Jacobi polynomials and

$$\lambda=\frac{1}{2}\left[\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(l-\Phi)\,(l-\Phi+1)+2\,M\,b}{\alpha^2}}\right)-\frac{\frac{M}{2\,\alpha^2\,\delta^2}\,(V_0+2\,a\,\delta)}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(l-\Phi)\,(l-\Phi+1)+2\,M\,b}{\alpha^2}}\right)}\right]. \tag{21}$$

Equation (19) is the non-relativistic bound-state energy levels and Eq. (20) is the radial wave functions of Schrödinger particles confined by the AB-flux field with superposed potential in a point-like global monopole defect. We can see that the eigenvalue solution is influenced by the topological defects of the geometry characterised by the parameter α and modified the result in comparison with the flat space case. Furthermore, the energy levels depend on the magnetic flux field and shifted more in addition to the topological defects, and this is a periodic function of the geometric quantum phase with a periodicity Φ_0. Thus, we have that $E_{n,l}(\Phi_{AB}\pm\Phi_0\,\nu)=E_{n,l\mp\nu}(\Phi_{AB})$, where $\nu=0,1,2,3,...$ This dependence of the energy levels on the magnetic flux field shows an analogue to the Aharonov-Bohm effect [57, 58] for the bound-state.

Now, we discuss below the effects of various factors one by one on the eigenvalue solution of the quantum system.

Case A: Without topological defects

We want to study the above quantum mechanical problem in absence of topological defects of the geometry. Therefore, for $\alpha\to 1$, the space-time (5) under consideration will become Minkowski flat space. Thus, for $\alpha\to 1$, the bound-state energy eigenvalue expression will be

$$E_{n,l}=-\frac{\delta^2}{2\,M}\left[\left(n+\frac{1}{2}+\sqrt{\left(l-\Phi+\frac{1}{2}\right)^2+2\,M\,b}\right)-\frac{\frac{M}{2\,\delta^2}\,(V_0+2\,a\,\delta)}{\left(n+\frac{1}{2}+\sqrt{\left(l-\Phi+\frac{1}{2}\right)^2+2\,M\,b}\right)}\right]^2. \tag{22}$$

The radial wave functions are given by

$$\psi_{n,l}(s) = s^{\zeta}\,(1-s)^{\frac{1}{2}+\sqrt{\left(l-\Phi+\frac{1}{2}\right)^2+2\,M\,b}}\,P_n^{\left(2\,\zeta\,,\,2\sqrt{\left(l-\Phi+\frac{1}{2}\right)^2+2\,M\,b}\right)}\,(1-2\,s), \tag{23}$$

where

$$\zeta = \frac{1}{2}\left[\left(n+\frac{1}{2}+\sqrt{\left(l-\Phi+\frac{1}{2}\right)^2+2\,M\,b}\right) - \frac{\frac{M}{2\,\delta^2}\,(V_0+2\,a\,\delta)}{\left(n+\frac{1}{2}+\sqrt{\left(l-\Phi+\frac{1}{2}\right)^2+2\,M\,b}\right)}\right]. \tag{24}$$

We can see that the bound-state energy levels (22) and the radial wave functions (23)–(24) of non-relativistic particles interact with Hulthen plus Yukawa and inverse quadratic potential in the flat space background get shifted or modified by the quantum flux field that shows an analogue of the Aharonov-Bohm effect [57, 58] for the bound-state.

Case B: Without magnetic flux field

In this case, we analyze the quantum mechanical problem in absence of magnetic flux field. Therefore, for $\Phi_{AB} \to 0$, the bound state energy eigenvalue from Eq. (19) will be

$$E_{n,l} = -\frac{\alpha^2\,\delta^2}{2\,M}\left[\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{l\,(l+1)+2\,M\,b}{\alpha^2}}\right) - \frac{\frac{M}{2\,\alpha^2\,\delta^2}\,(V_0+2\,a\,\delta)}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{l\,(l+1)+2\,M\,b}{\alpha^2}}\right)}\right]^2. \tag{25}$$

And the corresponding radial wave functions are given by

$$\psi_{n,l}(s) = s^{\vartheta}\,(1-s)^{\frac{1}{2}+\sqrt{\frac{l\,(l+1)+2\,M\,b}{\alpha^2}+\frac{1}{4}}}\,P_n^{\left(2\,\vartheta\,,\,2\sqrt{\frac{l\,(l+1)+2\,M\,b}{\alpha^2}+\frac{1}{4}}\right)}\,(1-2\,s), \tag{26}$$

where

$$\vartheta = \frac{1}{2}\left[\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{l\,(l+1)+2\,M\,b}{\alpha^2}}\right) - \frac{\frac{M}{2\,\alpha^2\,\delta^2}\,(V_0+2\,a\,\delta)}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{l\,(l+1)+2\,M\,b}{\alpha^2}}\right)}\right]. \tag{27}$$

One can see that only the topological defects of the geometry characterised by the parameter α shifted the bound state energy levels (25) and the radial wave functions (26)–(27) with this superposed potential in a point-like global monopole.

3. Applications to some individual and combined potential models

In this section, we discuss now the above quantum mechanical problem to some known individual potential as well as combined potential models and analyze the effects of the topological defect and the magnetic flux field. One can see that the eigenvalue solution gets modified by these factors with individual or combined potential.

3.1. Hulthen potential

The original Hulthen potential [9, 10] can be recovered by setting the parameters $\delta = 1/2\,\kappa$, $a = 0$, $b = 0$, and $V_0 = Z\,e^2/\kappa = K/\kappa$, $K = Z\,e^2$ in the general potential expression (10). We have obtained the following potential form [73]

$$V_H = -\frac{K}{\kappa}\,\frac{e^{-\frac{r}{\kappa}}}{(1-e^{-\frac{r}{\kappa}})}. \tag{28}$$

Thereby, using this potential (28) in the radial Eq. (9) and following a similar procedure, one can obtain the following bound state energy eigenvalue expression given by

$$E_{n,l} = -\frac{\alpha^2}{8\,\kappa^2\,M}\left[\left(n+\frac{1}{2}+\sqrt{\frac{(l-\Phi)\,(l-\Phi+1)}{\alpha^2}+\frac{1}{4}}\right) - \frac{\frac{2\,M\,K\,\kappa}{\alpha^2}}{\left(n+\frac{1}{2}+\sqrt{\frac{(l-\Phi)\,(l-\Phi+1)}{\alpha^2}+\frac{1}{4}}\right)}\right]^2. \tag{29}$$

The radial wave functions are given by

$$\psi_{n,l}(s) = s^{\iota}\,(1-s)^{\frac{1}{2}+\sqrt{\frac{(l-\Phi)\,(l-\Phi+1)}{\alpha^2}+\frac{1}{4}}}\;P_n^{\left(2\,\iota\;,\;2\sqrt{\frac{(l-\Phi)\,(l-\Phi+1)}{\alpha^2}+\frac{1}{4}}\right)}\,(1-2\,s), \tag{30}$$

where

$$\iota = \frac{1}{2}\left[\left(n+\frac{1}{2}+\sqrt{\frac{(l-\Phi)\,(l-\Phi+1)}{\alpha^2}+\frac{1}{4}}\right) - \frac{\frac{2\,M\,K\,\kappa}{\alpha^2}}{\left(n+\frac{1}{2}+\sqrt{\frac{(l-\Phi)\,(l-\Phi+1)}{\alpha^2}+\frac{1}{4}}\right)}\right]. \tag{31}$$

Equation (29) is the bound-state energy levels and Eq. (30)–(31) is the radial wave function of a non-relativistic particle under AB-flux field with Hulthen potential in point-like global monopole. This eigenvalue solution gets modified in comparison to those results obtained in Refs. [11,12,73] in flat space background due to the presence of the topological defects characterised by the parameter α, and the magnetic flux field Φ_{AB} considered in the quantum system. One can see that the energy levels depend on the magnetic flux field and this dependence of the energy levels on the geometric quantum phase gives us an analogue of the Aharonov-Bohm effect [57, 58] for the bound-state.

If we analyze this quantum system without magnetic flux field, that is, $\Phi_{AB} \to 0$, the eigenvalue solution from Eqs. (29)–(31) becomes

$$E_{n,l} = -\frac{\alpha^2}{8\,\kappa^2\,M}\left[\left(n+\frac{1}{2}+\sqrt{\frac{l\,(l+1)}{\alpha^2}+\frac{1}{4}}\right) - \frac{\frac{2\,M\,K\,\kappa}{\alpha^2}}{\left(n+\frac{1}{2}+\sqrt{\frac{l\,(l+1)}{\alpha^2}+\frac{1}{4}}\right)}\right]^2,$$

$$\psi_{n,l}(s) = s^{\varsigma}\,(1-s)^{\frac{1}{2}+\sqrt{\frac{l\,(l+1)}{\alpha^2}+\frac{1}{4}}}\;P_n^{\left(2\,\varsigma\,,\,2\sqrt{\frac{l\,(l+1)}{\alpha^2}+\frac{1}{4}}\right)}\,(1-2\,s), \tag{32}$$

where

$$\varsigma = \frac{1}{2}\left[\left(n+\frac{1}{2}+\sqrt{\frac{l\,(l+1)}{\alpha^2}+\frac{1}{4}}\right) - \frac{\frac{2\,M\,K\,\kappa}{\alpha^2}}{\left(n+\frac{1}{2}+\sqrt{\frac{l\,(l+1)}{\alpha^2}+\frac{1}{4}}\right)}\right]. \tag{33}$$

One can see that only the topological defect of point-like global monopole influences the bound-state eigenvalue solution of a non-relativistic particle with Hulthen potential. The global effects of the geometry characterised by the parameter α is present explicitly on the eigenvalue solution which modified the result.

On the other hand, if we analyze the quantum system without topological defects, that is, $\alpha \to 1$, the space-time geometry (5) under consideration will become Minkowski flat space. Therefore, for $\alpha \to 1$, the eigenvalue solution from Eqs. (29)–(31) becomes

$$E_{n,l} = -\frac{1}{2\,M}\left[\frac{(n+l-\Phi+1)}{2\,\kappa} - \frac{M\,K}{(n+l-\Phi+1)}\right]^2,$$

$$\psi_{n,l}(s) = s^{\varpi}\,(1-s)^{l-\Phi+1}\,P_n^{(2\,\varpi\,,\,2\,(l-\Phi)+1)}\,(1-2\,s), \tag{34}$$

where $\varpi = 1/2\left[(n+l-\Phi+1) - (2\,M\,K\,\kappa/(n+l-\Phi+1))\right]$.

Equation (34) is the bound-state eigenvalue solution of a non-relativistic particle confined by the AB-flux field with Hulthen potential in the flat space background. Note that for zero magnetic flux field, $\Phi_{AB} \to 0$, this eigenvalue solution (34) reduces to the result obtained in Ref. [73]. Thus, we can see that the presence of the magnetic flux field in the quantum system shifted the energy levels and the radial wave functions which shows an analogue of the Aharonov-Bohm effect for the bound-state [57,58].

3.2. Hulthen-Yukawa potential

In this section, we set the parameter $b = 0$ in the potential (6), one will have a combined potential called Hulthen-Yukawa potential (HYP) given by

$$V(r) = -V_0\,\frac{e^{-2\,\delta\,r}}{1-e^{-2\,\delta\,r}} - \frac{a}{r}\,e^{-\delta\,r}. \tag{35}$$

Thereby using this combined potential in the radial equation (12) and following the previous procedure, one can find the following bound-state energy eigenvalue expression given by

$$E_{n,l}=-\frac{\alpha^2\,\delta^2}{2\,M}\left[\left(n+\frac{1}{2}+\sqrt{\frac{(l-\Phi)\,(l-\Phi+1)}{\alpha^2}+\frac{1}{4}}\right)-\frac{\frac{M}{2\,\alpha^2\,\delta^2}\,(V_0+2\,a\,\delta)}{\left(n+\frac{1}{2}+\sqrt{\frac{(l-\Phi)\,(l-\Phi+1)}{\alpha^2}+\frac{1}{4}}\right)}\right]^2. \tag{36}$$

The radial wave functions are given by

$$\psi_{n,l}(s)=s^{\sigma}\,(1-s)^{\frac{1}{2}+\sqrt{\frac{(l-\Phi)\,(l-\Phi+1)}{\alpha^2}+\frac{1}{4}}}\,P_n^{\left(2\,\sigma,2\,\sqrt{\frac{(l-\Phi)\,(l-\Phi+1)}{\alpha^2}+\frac{1}{4}}\right)}\,(1-2\,s), \tag{37}$$

where

$$\sigma=\frac{1}{2}\left[\left(n+\frac{1}{2}+\sqrt{\frac{(l-\Phi)\,(l-\Phi+1)}{\alpha^2}+\frac{1}{4}}\right)-\frac{\frac{M}{2\,\alpha^2\,\delta^2}\,(V_0+2\,a\,\delta)}{\left(n+\frac{1}{2}+\sqrt{\frac{(l-\Phi)\,(l-\Phi+1)}{\alpha^2}+\frac{1}{4}}\right)}\right].$$

Equation (36) is the non-relativistic bound-state energy levels and Eq. (37) is the radial wave function of a Schrodinger particle confined by the AB-flux field with superposed Hulthen-Yukawa potential in a point-like global monopole. This eigenvalue solution gets modified by the topological defects of point-like global monopole characterised by the parameter α, and the magnetic flux field.

In absence of magnetic flux field, that is, $\Phi_{AB}\to 0$, the eigenvalue solution from Eqs. (36)–(37) becomes

$$E_{n,l}=--\frac{\alpha^2\,\delta^2}{2\,M}\left[\left(n+\frac{1}{2}+\sqrt{\frac{l\,(l+1)}{\alpha^2}+\frac{1}{4}}\right)-\frac{\frac{M}{2\,\alpha^2\,\delta^2}\,(V_0+2\,a\,\delta)}{\left(n+\frac{1}{2}+\sqrt{\frac{l\,(l+1)}{\alpha^2}+\frac{1}{4}}\right)}\right]^2,$$

$$\psi_{n,l}(s)=s^{\Delta}\,(1-s)^{\frac{1}{2}+\sqrt{\frac{l\,(l+1)}{\alpha^2}+\frac{1}{4}}}\,P_n^{\left(2\,\Delta,2\,\sqrt{\frac{l\,(l+1)}{\alpha^2}+\frac{1}{4}}\right)}\,(1-2\,s), \tag{38}$$

where

$$\Delta=\frac{1}{2}\left[\left(n+\frac{1}{2}+\sqrt{\frac{l\,(l+1)}{\alpha^2}+\frac{1}{4}}\right)-\frac{\frac{M}{2\,\alpha^2\,\delta^2}\,(V_0+2\,a\,\delta)}{\left(n+\frac{1}{2}+\sqrt{\frac{l\,(l+1)}{\alpha^2}+\frac{1}{4}}\right)}\right]. \tag{39}$$

We can see that the eigenvalue solution (38) is only influenced by the topological defects of the geometry characterised by the parameter α which modified the result in comparison to flat space case with this combined potential.

On the other hand, if we analyze the quantum system without topological defects, that is, $\alpha\to 1$, the space-time geometry will become Minkowski flat space. Therefore, for $\alpha\to 1$, the bound-state eigenvalue solution will be

$$E_{n,l}=-\frac{\delta^2}{2\,M}\left[(n+l-\Phi+1)-\frac{\frac{M}{2\,\delta^2}\,(V_0+2\,a\,\delta)}{(n+l-\Phi+1)}\right]^2, \tag{40}$$

The radial wave functions are given by

$$\psi_{n,l}(s)=s^{\varphi}\,(1-s)^{l-\Phi+1}\,P_n^{(2\,\varphi,2\,(l-\Phi)+1)}\,(1-2\,s), \tag{41}$$

where

$$\varphi=\frac{1}{2}\left[(n+l-\Phi+1)-\frac{\frac{M}{2\,\delta^2}\,(V_0+2\,a\,\delta)}{(n+l-\Phi+1)}\right].$$

Equations (40)-(41) is the bound-state eigenvalue solution of a non-relativistic particle under the influence of the AB-flux field with Hulthen plus Yukawa potential in the flat space background. One can see that the energy levels depend on the magnetic flux which shows an analogue of the Aharonov-Bohm effect for the bound-state [57, 58].

3.3. Hulthen-Inverse Quadratic Potential

In this section, we set the parameters $a =$, $V_0 = Z\,e^2/\kappa$, $\delta = 1/2\,\kappa$ in potential (10), we have

$$V(r) = -\frac{K}{\kappa}\,\frac{e^{-\frac{r}{\kappa}}}{1-e^{-\frac{r}{\kappa}}} + \frac{b}{r^2}. \tag{42}$$

Thereby, substituting this combined potential (42) in the radial equation (9) and following the same procedure, one can obtain the following eigenvalue solution

$$E_{n,l} = -\frac{\alpha^2\,\delta^2}{2\,M}\left[\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(l-\Phi)\,(l-\Phi+1)+2\,M\,b}{\alpha^2}}\right) - \frac{\frac{2\,M\,K\,\kappa}{\alpha^2}}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(l-\Phi)\,(l-\Phi+1)+2\,M\,b}{\alpha^2}}\right)}\right]^2,$$

$$\psi_{n,l}(s) = s^{\chi}\,(1-s)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(l-\Phi)\,(l-\Phi+1)+2\,M\,b}{\alpha^2}}}\;P_n^{\left(2\,\chi,2\,\sqrt{\frac{1}{4}+\frac{(l-\Phi)\,(l-\Phi+1)+2\,M\,b}{\alpha^2}}\right)}\,(1-2\,s). \tag{43}$$

where

$$\chi = \frac{1}{2}\left[\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(l-\Phi)\,(l-\Phi+1)+2\,M\,b}{\alpha^2}}\right) - \frac{\frac{2\,M\,K\,\kappa}{\alpha^2}}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(l-\Phi)\,(l-\Phi+1)+2\,M\,b}{\alpha^2}}\right)}\right]. \tag{44}$$

Equation (43) is the eigenvalue solution of a non-relativistic particle confined by the AB-flux field in point-like global monopole with this combined Hulthen-Inverse Quadratic potential. One can see that the topological defects characterised by the parameter α modified the eigenvalue solution in comparison with the flat space result.

If we analyze the quantum system without topological defects, that is, $\alpha \to 1$, the space-time will become Minkowski flat space. Therefore, for $\alpha \to 1$, the eigenvalue solution becomes

$$E_{n,l} = -\frac{\delta^2}{2\,M}\left[\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+(l-\Phi)\,(l-\Phi+1)+2\,M\,b}\right) - \frac{2\,M\,K\,\kappa}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+(l-\Phi)\,(l-\Phi+1)+2\,M\,b}\right)}\right]^2,$$

$$\psi_{n,l}(s) = s^{\chi_1}\,(1-s)^{\frac{1}{2}+\sqrt{\frac{1}{4}+(l-\Phi)\,(l-\Phi+1)+2\,M\,b}}\;P_n^{\left(2\,\chi_1,2\,\sqrt{\frac{1}{4}+(l-\Phi)\,(l-\Phi+1)+2\,M\,b}\right)}\,(1-2\,s). \tag{45}$$

where

$$\chi_1 = \frac{1}{2}\left[\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+(l-\Phi)\,(l-\Phi+1)+2\,M\,b}\right) - \frac{2\,M\,K\,\kappa}{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+(l-\Phi)\,(l-\Phi+1)+2\,M\,b}\right)}\right]. \tag{46}$$

Thus, we can see that the magnetic flux field shifts the eigenvalue solution and one can observe an analogous of the Aharonov-Bohm effect for the bound-state.

4. Conclusions

To sum up, in this paper, we have investigated the approximate eigenvalue solutions of the three-dimensional radial Schrödinger equation in the presence of Aharonov-Bohm flux field with potential under topological effects produced by a point-like global monopole. The presence of topological defects (cosmic strings, global monopoles) makes the space-time geometry curved and changes the physical properties of a quantum system. The studies of the wave equations in curved space-time with topological defects have physical importance and significance. The space-time geometry under consideration in this analysis possesses a curvature singularity on the axis and reduces to Minkowski flat space for $\alpha \to 1$. We have derived the radial wave equation of the Schrödinger equation in a curved geometry via a point-like global monopole with potential $V(r)$. Then, in Sec. 2, we have chosen a potential superposed of Hulthen, Yukawa and inverse quadratic potentials that has many applications in different branches of physics and chemistry. We employed the Greene-Aldrich improved approximation scheme into the centrifugal and reciprocal terms that appeared in the radial equation and arrived at a second-order homogeneous differential equation after a suitable transformation. This equation was then solved using the parametric NU-method and the eigenvalue solution was obtained. The energy levels given by Eq. (19) and the radial functions by Eq. (20) of the non-relativistic particles. One can see that the topological defect parameter characterised by α and the magnetic flux field Φ_{AB} shifts the eigenvalue solution and modified them in comparison to the flat space result with this superposed potential. Furthermore, one can see that the eigenvalue solution depends on the magnetic flux field, and this dependence of the eigenvalue solution on the geometric quantum phase shows an analogue to the Aharonov-Bohm effect [57, 58] for the bound-state.

In Sec. 3, we utilized the above eigenvalue solution for individual and some combined potential models and analyzed the results. For example, in Subsec. 3.1, we used Hulthen potential only and the eigenvalue solution (29)–(30) was obtained using a similar procedure done earlier. In Subsec. 3.2, Hulthen plus Yukawa potential was used and the bound-state eigenvalue solution (36)–(37) was obtained. In Subsec. 3.3, Hulthen plus inverse quadratic potential was used and the bound state eigenvalue solution (43) was obtained. In all cases, we have seen that the topological defect represented by the parameter α and the magnetic flux Φ shifted the eigenvalue solutions and modified them in comparison to the flat space results with these potential models.

Thus, we investigated the quantum motions of non-relativistic particles confined by the Aharonov-Bohm flux field with potential in a point-like defect. We verified that the global effect of the geometry represented by the parameter α was present explicitly in the energy levels and the structure of states. The presence of topological defect shifted the eigenvalue solutions in comparison with the flat space results with the chosen potential and broke the degeneracy. For $\alpha \to 1$, the eigenvalue solution reduces to flat space result with these potential models. It is worth mentioning that several authors studied the non-relativistic wave equation with different combined potential models in flat space background. In the present analysis, we have studied the quantum mechanical problem in a topological defect produced by a point-like global monopole. We believe that the presented results are interesting and have significance in the literature.

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Conflict of Interest

There is no conflict of interests regarding publication of this paper.

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