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Accurate delta hedging of european options using conformable calculus

Cobertura delta precisa de opciones europeas usando cálculo conformable

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Abstract

Objective: we aim to develop a method for delta hedging portfolios of European options based on the theory of conformable calculus which improves accuracy of risk management of listed options in a first-order approximation.

Methodology: we allow the time derivative in the classic Black-Scholes-Merton model to have a fractional order $0 \leq \alpha \leq 1$ and calculate the corresponding delta of a portfolio of listed options as a function of this conformable parameter.

Results: applying this method to a portfolio consisting of eight European options on the SPX index, we find that conformable delta hedging offers more accurate average predictions than classical delta hedging.

Limitations: this method is applicable for delta hedging in European options only.

Originality: this is the first successful application of conformable calculus to delta hedging in European options.

Conclusions: application of Conformable Calculus allows for a greater flexibility in the local approximation to price in delta-hedging European options and offers a new and more precise methodology to this objective.

Keywords: option pricing, delta hedging, conformable calculus, risk management.

JEL Classification: G12, G17, G19.

Resumen

Objetivo: desarrollar un método para la cobertura delta de portafolios de opciones europeas listadas con base en la teoría del cálculo conformable que mejora la precisión de las predicciones usando la aproximación de primer orden.

Metodología: permitimos que la primera derivada en el modelo clásico de Black-Scholes-Merton tenga un orden fraccional $0 \leq \alpha \leq 1$ y calculamos la delta correspondiente de un portafolio como función de este parámetro conformable.

Resultados: aplicando este método a un portafolio conformado de ocho opciones europeas listadas sobre el índice SPX, encontramos que la cobertura conformable genera predicciones más precisas, en promedio, que la cobertura tradicional.

Limitaciones: este método es aplicable solamente a la cobertura delta (hedging) de opciones europeas.

Originalidad: esta es la primera aplicación exitosa del cálculo conformable a la cobertura delta en opciones europeas.

Conclusiones: la aplicación del cálculo conformable permite mayor flexibilidad en la aproximación local implícita en la cobertura delta de portafolios de acciones europeas y se ofrece como una metodología novel y de mayor precisión que la tradicional.

Palabras clave: precio de opciones, cobertura delta, cálculo conformable, gestión de riesgos.

Clasificación JEL: G12, G17, G19.

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Introduction

The effective management of risk within a portfolio of options stands as a cornerstone in financial mathematics, garnering extensive scholarly attention over the years. Among the myriad of advancements, the Black-Scholes-Merton model (BSM), Black and Scholes (1973), Merton (1973), has emerged as a pivotal landmark, offering insights into the intricate world of options. This, in turn, paved the way for the development of delta hedging, an elementary yet indispensable technique outlined in Hull (2017), and its variations such as Hull and White (1987).

However, the financial landscape has undergone a profound transformation since the tumultuous events of the 2007-08 global financial crisis. The ensuing years have witnessed a paucity of innovations in risk management strategies, particularly those concerning the enhancement and robustness of existing methodologies. These methodologies have yet to be stress-tested against various challenges, including capital market withdrawals, demographic shifts, de-globalization, the rise of populism, and wealth redistribution via unconventional means such as "helicopter money".

Traditional delta hedging is based on the analysis of the sensitivity of an option's price to changes in the price of the underlying assets. This is done using calculus; that is, computing the first derivative of the portfolio price with respect to the underlying asset's price. More formally, let V_t be the value of an option at time *t* and S_t be the price of the underlying asset at the same point in time. Write:

$$V_t = f(S_t, Z_t) \tag{1}$$

Where Z_t contains other, possibly stochastic, factors that affect option prices such as volatility or the risk–free interest rate. The quantity of interest is, then dV_t/dS_t as it measures deviations of the option price per unit changes in the underlying asset's price.

Under the assumptions of Black and Scholes (1973) and Merton (1973), the sensitivity of (1) is the basis of delta hedging. However, as explained by Hull and White (2017), among others, this hedging strategy is, at best, approximate, since it relies on very stringent assumptions.

By far, the most contested hypothesis of this classical model is homoscedasticity, *i.e.*, constant volatility. For this reason, some authors like Hull and White (2017) or Xia *et al.* (2023) have studied ways to adjust the computed sensitivity to improve hedging. While the proposed solutions are mostly based on financial considerations, we propose an alternative based on mathematics. To this end, we use the conformable derivative. Conformable calculus, initially conceptualized by Khalil *et al.* (2014) and Abdeljawad (2015), has gained increasing attention for its versatility and applicability across diverse fields, ranging from physics and mathematics to engineering, as comprehensively reviewed by Chung (2015) and Zhou *et al.* (2018). Despite its potential, the applications of conformable calculus to finance are scarce. In particular, the literature of derivatives and pricing does not use this kind of calculus and is mainly based on integer-order considerations.

This paper intends to close this gap by providing a first study into the viability of using conformable calculus in the context of pricing European options. Specifically, the Black-Scholes model implies a price function that can be approximated by a first or second-order Taylor series. These approximations, that are used commonly by practitioners, are at the basis of the so-called Delta-Hedging method. Broadly speaking, the approximated price is used to gauge potential price movements of a portfolio in the close future and this measure is, in turn, used to tailor an ad-hoc financial derivative to hedge risk exposure. The

Taylor series under consideration is based in integer-order calculus and provides an acceptable local approximation to price movements, but its accuracy may diminish depending on several factors such as, most notably, correlation between the price of the underlying asset and its volatility. This way, in periods of stable market prices and stable volatility, the approximated prices are accurate enough; but with sudden increases in volatility or changing market expectations, the forecasts may lose in precision in the short term. In general, since volatility is a stationary stochastic process, traditional forecasts are expected to catch up with these market changes in as much as these periods are followed by a return to stability. However, during the period of agitation, volatility and transition, errors could be costly. With the intention to alleviate these problems, we propose a modification of the classical Greeks implied by the Black-Scholes model, which in turn provides an alternative Taylor approximation to option prices, using conformable calculus, and then evaluate the usefulness of the strategy in terms of forecasting accuracy. That is, we introduce conformable calculus as a tool to address the intricacies of Delta Hedging in modern financial markets and evaluate its usefulness in a forecasting environment.

The rest of the paper is structured as follows. In **Conformable calculus and delta hedging** we lay the foundations by presenting the conformable derivative and its defining properties. Subsequently, we apply these principles to the challenge of hedging a financial portfolio, offering a novel perspective on risk management. In **Empirical Analysis** we substantiate our theoretical framework with empirical evidence, shedding light on the practical implications and performance of our approach. Finally, we draw our conclusions and provide food for thought in our closing remarks. Through this endeavor, we contribute to the evolving landscape of risk management in finance, thus expanding the horizons of the applicability of conformable calculus within this crucial domain of financial practice.

Conformable calculus and delta hedging

Fractional calculus, meaning calculus of a non-integer order, has attracted the attention of many researchers in various fields due to its effectiveness in modeling general nonlinear systems. In this section we present some basic facts on Conformable Derivatives and its applications to delta hedging. Several definitions have been put forward for a fractional derivative operator, like those by L'Hôpital, Riemann-Liouville, and Caputo. Unfortunately, these definitions do not satisfy some basic properties of the traditional, integer-order, derivative operator like the chain rule or the product rule. See Kilbas *et al.* (2006) for a review of the properties of fractional operators. The relevance of fractional calculus stems from the wide array of applications in physics and engineering like fractional conservation of mass, fractional electrochemical analysis, groundwater flow problems with fractional operators, fractional advection-dispersion equation, time-space fractional diffusion equation models, acoustic wave equations for complex media, and the fractional Schrödinger equation in quantum theory, among others.

A new definition of fractional derivatives in put forward in Khalil et al. (2014) and Abdeljawad (2015) which has given rise to the so-called conformable derivative. Given a function $f\colon[0,\infty)\to\mathbb{R}$, its conformable derivative of order $\alpha\in(0,1)$ is defined for all $t>0$ by the limit

$$T_\alpha(f)(t)=\lim_{\epsilon\to\infty}\left(\frac{f(t+\epsilon t^{1-\alpha})-f(t)}{\epsilon}\right)$$

Although the limit is clearly different from the one appearing in the classical definition of derivative, it is shown in Anderson and Camrud (2019) that if f is differentiable (in the classical sense), then:

$$T_\alpha(f)(t) = t^{1-\alpha}\frac{df(t)}{dt} \quad (2)$$

Thus, in a sense, the conformable derivative is a local homotopic expansion of the classical derivative via the term $t^{1-\alpha}$ which adds a degree of complexity to the differential approximation of the first order of f. Viewed otherwise, it is a change of variables via $u = \frac{x^\alpha}{\alpha}$ for differentiable functions *f*.

This fractional derivative is not only easily defined, but also preserves some of the most useful properties of the classical derivative operator such as a version of the chain rule and an expansion in the Taylor series. See, for instance Abdeljawad (2015). It is also a local operator, as shown in Anderson and Ulness (2015), Anderson and Camrud (2019). For these reasons, among others, this conformable operator has been widely used in applications in diverse fields. For instance, Martynyuk et. al. (2019), Martynyuk and Stamova (2018), Martynyuk and Timoshenko (2018), Zhao and Luo (2017), Anderson & Camrud (2018) and El-Ajou (2020) give a physical and geometrical interpretation of the conformable derivative which makes it possible to apply it to problems in physics and engineering.

In finance, however, applications of conformable calculus and risk modelling are scarce. Any starting trader or market-maker in any options contract will need to manage its exposure to risk, whether the option is listed on an exchange or is an OTC contract. Without a proper hedging scheme in place, an adverse price movement has the potential of bankrupting a trader. There are many ways of managing risk. An intuitive approach is to express the change of the option price V_t as a function of a change in one of the inputs used for calculating its value: the price of the underlying asset (S_t), its volatility (σ_t), the risk–free interest rate(r_t), and the dividend yield(q_t). As explained in Hull (2017) a market-maker is interested in changes in the value of an option given a change in one of the inputs, *ceteris paribus*.

This makes calculus a most intuitive tool. In fact, one of the most studied quantities in risk management is the so-called delta of an option of portfolio which is nothing but

$$\Delta = \frac{\partial V}{\partial S} \quad (3)$$

the (*ceteris paribus*) change in the price of the option given a unit change in the price of the underlying asset.

In fact, traders and market-makers can control risk exposure through a simple procedure called Delta-Hedging (D-H), which is at the base of more complex hedging strategies. In essence D-H allows the trader to replicate the payoff of an option by taking an offsetting position in shares of the underlying asset. This way, the trader uses a self-financing strategy whose value at maturity is exactly the value of the option that is being replicated.

In this paper, we analyze the conformable derivative of the price of the option as a function of the price of the underlying asset. Thus, instead of using the classic delta in formula (3), we analyze

$$\Delta_{C,\alpha} = T_\alpha(V_t)(S_t),$$

the conformable delta. To justify this choice, it is worth remembering where the classical delta comes from and why it is so widely used.

According to the Black-Scholes-Merton model to price options, an industry standard, the underlying stock prices follow a geometric Brownian motion, that is

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Where W_t is a standard Brownian motion.

This, in turn and via Itô's Lemma, leads to the well-known differential equation

$$\frac{\partial V}{\partial t} + r_f S \frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 = r_f V, \quad (4)$$

for the price of an option V_t. If we let K be the strike price, τ be the time to maturity, r_f be the risk–free interest rate, and q be the dividend yield of an option, we can solve these equations for a call-and-put, respectively, obtaining the theoretical prices

$$C = S_t e^{-q\tau} \mathcal{N}(d_1) - K e^{-r_f \tau} \mathcal{N}(d_2), \quad (4b)$$

$$P = K e^{-r_f \tau} \mathcal{N}(-d_2) - S_t e^{-q\tau} \mathcal{N}(-d_1), \quad (4c)$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r_f - q + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \text{ and} \quad (4d)$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

Classical calculus now gives us

$$\Delta = \frac{\partial V}{\partial S} = e^{-q\tau}\Phi(d_1) \quad (5)$$

with Φ(·) the standard normal distribution function. Debates around the use of this model are ubiquitous. As is well known, it assumes no transaction costs, the possibility of short sales and continuous trading, and the implicit ability to borrow and lend at the risk–free interest rate. Given that these assumptions are not entirely feasible, that there always are transaction costs, that continuous trading is materially impossible, and that the risk-free interest rate is more of a gauge than a real financial product for most users, delta hedging based on this model is only approximate.

From a purely mathematical point of view, the inadequacy of the assumptions to real life trading implies that the local approximation provided by the first order Taylor series to the value of the option is seldom inaccurate. A possible solution for this kind of situation in other approximation contexts is to provide this approximation with some information about curvature. In this paper, we use the conformable derivative as a means to provide this curvature and improve the accuracy of the first order, or linear, approximation to the value of the option.

In order to translate this into a conformable derivative analysis, we first let the time derivative in the Black-Scholes equation (4) to be conformable of order $0 < \alpha < 1$. Using the results in Anderson & Camrud (2018), this amounts to solving the differential equation subject to the change of variables $u = \frac{x^\alpha}{\alpha}$. This in turn implies that we can use equation (2) to compute the conformable derivative of the option's price with respect to S in the original Black-Scholes solution in equations (4b) and (4c). Denoting by Δ_α the conformable Delta of the option, and starting with a Call, we see that

$$\Delta_\alpha = T_\alpha(V_t)(S_t) = S^{1-\alpha}\frac{\partial C(V,t)}{\partial S} = S^{1-\alpha} e^{-q\tau}\Phi(d_1) \quad (6)$$

The calculation for the put option is analogous. It is these expressions that we use for delta hedging. In order to test the efficacy of this method, we calculate the value of the replicating portfolio using both the classical delta hedging method and the conformable delta hedging method. To this end, we partition the interval [0,τ] as

$$t_0 = 0 < t_1 < \cdots < t_n = \tau$$

for a given natural *n*. The partition is assumed regular, so that $t_{i+1} = t_i + \delta_t$. To keep the notation simple, let V_i and Δ_i be the value of the option and its delta at time t_i. The hedging method approximates the value of the option at time t_{i+1} as

$$V_{i+1} = V_i + \Delta_i\left(S_{i+1} - S_i(1 - q\delta_t)\right) \quad (7)$$

On the other hand, and in an analogous way, letting $\Delta_{\alpha,i}$ be the value of the conformable delta at time t_i, the first order conformable linear approximation is given by

$$V_{i+1} = V_i + \Delta_{\alpha,i}\left(S_{i+1} - S_i\left(1 - q\delta_t\right)\right) \quad (8)$$

Observe that the correction to the predicted prices will only come from the curvature implied by the term $S_{t_i}^{1-\alpha}$ in equation (6). An intuitive conclusion is that α will necessarily be a number close to 1 (otherwise the correction would be too strong).

Empirical Analysis

The methods developed in the previous section provide us with a new approach to delta hedging based on conformable calculus. Intuitively, a better fit for real price movements is expected due to the implicit use of a new source of curvature that we can control via a simple, real parameter. That is, the first-order conformable Taylor series for the price function at any time t is expected to predict future prices more accurately than the usual Taylor series.

To establish this empirically, we consider a portfolio of calls and puts all having the S&P500 index as the underlying asset (SPX index). Specifically, and in the spirit of the original CBOE Market Volatility Index (VIX), we consider a portfolio consisting of eight exchanged European index options: 4 calls and 4 puts. We choose 2 calls and 2 puts with a nearby expiration date and the remaining 4 options with a slightly larger expiration date. For the options with nearby expiration, we use a horizon of 30 days while for the second nearby expiration we use 37 days. Our portfolio is conformed in equal parts by all these options so that the value of our portfolio at time t is just the arithmetic average of the 8 options, namely

$$V_t = \frac{1}{4}\sum_{i=1}^{4} V_{i,t}$$

In both horizons, we include one call and one put at the money, and one call and one put out of the money. For out-of-the-money calls, the strike price considered is 102.5% of the spot whereas for out-of-the-money puts the strike price is 95% of the spot. In other words, one of the calls has a strike price equal, or very similar, to the actual price of the S&P500 (at the money) and the other one has a strike price in which losses would be incurred being 2.5% above the actual price of the S&P500. Similarly, one put has a strike price equal or very similar to the actual price of the S&P500 and the other one has a price 5% lower which would, thus, incur in loss.

The data used in this application were downloaded from the Bloomberg Professional Service, Bloomberg (2022), and include expiration dates starting on May 2022 and ending in November 2022. We begin valuing our portfolio on the initial date T_0= 2022/04/06 and roll it over every week to new contract maturities. That is, the assets with expiration horizon of 37 days are now considered nearby expiration and, thus, kept in the portfolio, and the assets with expiration horizon of 30 days are sold and replaced with new assets with an expiration date of 37 days. In other words, the four nearby expiring options are sold and thus taken out of the portfolio, and the four second nearby expiring are kept and now considered nearby expiring. Four new options are incorporated, all with an expiration of 37 days. We continued this process until 2022/09/28.

For every day in the period under analysis, we approximate the value of the underlying portfolio using formulas (7) and (8) for each option. Since the total value of the portfolio is just the average of these individual prices, we use the average of the forecast prices as the forecast for the portfolio prices. As we do this on a daily basis, when rebalancing at time t_i with maturity at time *T*, we use $\tau = T - t_i$ as the remaining time to maturity. In most evaluations the partition mesh is $\delta_t = 1$ with only two exceptions. One is when a portfolio is predicted over the weekend, in which case $\delta_t = 3$ is used. The other is when the portfolio is marked to market through a holiday. In this case δ_t is increased by one for each holiday. Using information from the Bloomberg Professional service, Bloomberg (2022), we take q=1.55% as an estimate of the dividend yield for the S&P500 index during the last 10 years.

At each time, therefore, we produce two sequences of forecasts, namely a classic one $\widehat{V}_{t+1}$ averaging formula (7) and a conformable one $\widehat{V}_{\alpha,t+1}$ averaging formula (8) instead. Comparing each sequence with the actual market price of the portfolio $\widehat{V}_{t+1}$, and following common practices in the field of forecast evaluation, we compute the following measures of accuracy as a function of the conformable parameter α:

$$MSE = \frac{1}{n}\sum_{t=1}^{n}\left(\widehat{V}_{\alpha,t} - V_t\right)^2$$

$$MAE = \frac{1}{n}\sum_{t=1}^{n}\left|\widehat{V}_{\alpha,t} - V_t\right|$$

$$MAPE = \frac{1}{n}\sum_{t=1}^{n}\left|\frac{\widehat{V}_{\alpha,t} - V_t}{V_t}\right|$$

The error measures are shown in **Figure 1**. A convex shape of all the measures shows that some accuracy can be gained in forecasting option prices using conformable delta hedging. As expected, mean squared error and mean absolute errors prescribe different optimal values for α, but they are very similar. On the other hand, mean absolute percentage error is an increasing function of α, making any of the chosen conformable parameters a better choice than the classical delta hedging method.

It is to be observed that the delta for the option using the derivative of integer order corresponds to the choice $\alpha = 1$. It can be observed in all the figures that the error to which the forecast is subject in the classic, integer order case, can be reduced using a conformable derivative operator. In fact, the convexity of all the error functions shows that the slight non-linearity induced by the term $S^{1-\alpha}$ in equation (6) allows for a more accurate forecasting of prices and thus a more precise hedging. To understand these better, the mean squared error and its decomposition into bias, variance, and irregular components is shown in **Figure 2**. Clearly, bias is a decreasing function of parameter α, which in turn implies that the classical integer-order delta provides the best hedging in terms of bias, that is, it minimizes bias.

However, the variance component presents the opposite behavior: it is an increasing function or the conformable parameter α. This implies that the forecast based on the traditional Black-Scholes model is more sensitive to volatility and less able to capture its implications than the one based on a fractional derivative of order $\alpha < 1$. Likewise, the irregular component is also seen to be better explained using conformable calculus than it is using the classical integer order solution to the Black-Scholes equations. The addition of these effects is what makes hedging based on conformable delta calculations a more accurate method overall.

Conclusions

As explained in the Introduction, conformable calculus has been successfully applied in different areas of physics and engineering providing a more flexible way to approximate functions to the first order. In finance, however, these applications are limited and, as far as we know, none exists for portfolio hedging.

As we have shown, the curvature added to the conformable delta via the term $S^{1-\alpha}$ helps re-dimension the rebalancing of the portfolio and increases accuracy. Of course, this accuracy comes at the cost of a new parameter. The estimation of this parameter, however, is easily implemented by minimizing rebalancing error in a training set of option prices.

In order to give the conformable delta hedging a financial interpretation, we turn back to our findings: conformable delta hedging captures volatility better. This is, indeed, one of the building blocks of alternative valuing formulas with increased precision. For instance, the use of the minimum variance delta of Hull and White (2017) helps increasing hedging accuracy because it accounts for the correlation between the price of the underlying asset and the other determinants of the option price, such as volatility. As we have shown, the conformable delta captures this correlation precisely, albeit not in an explicit manner. In fact, as parameter α moves away from its traditional value of 1, this correlation helps make the forecast more precise by adding a bit of curvature. More explicitly, the conformable derivative adds the term $S^{1-\alpha}$ to the usual delta as in equation (6) which helps introduce the element of price volatility into the hedging device. Moreover, since $\alpha \cong 1$, the changes produced by this device are minor whenever the price is relatively stable, but are more significant as changes are more abrupt.

Finally, we chose options based on the S&P500 because they represent the market as a whole. Thus, the findings in this paper are expected to be true when analyzing liquid options in mature markets. It is essential that the option be liquid; that is, that there is effective trade in the market for the option. Otherwise, the prices to feed the forecasting process are inaccessible. It is also essential for the market to be rich enough in products and trading so that prices obey the non-arbitrage conditions underlying the Black-Scholes model. Basically, our finding is that the deviations of the forecasts induced by the correlations not accounted for in the Black-Scholes model can be mitigated using conformable calculus instead of the classic integer-order approximations. The particular choice of the parameter value α has to be based on the data under analysis and will certainly vary from one application to another.

Figure 1

Mean squared error (MSE), mean absolute error (MAE), and mean absolute percentage error (MAPE) of the forecasted prices as a function of the conformable parameter α. The choice α = 1 corresponds to the classical BSM model.

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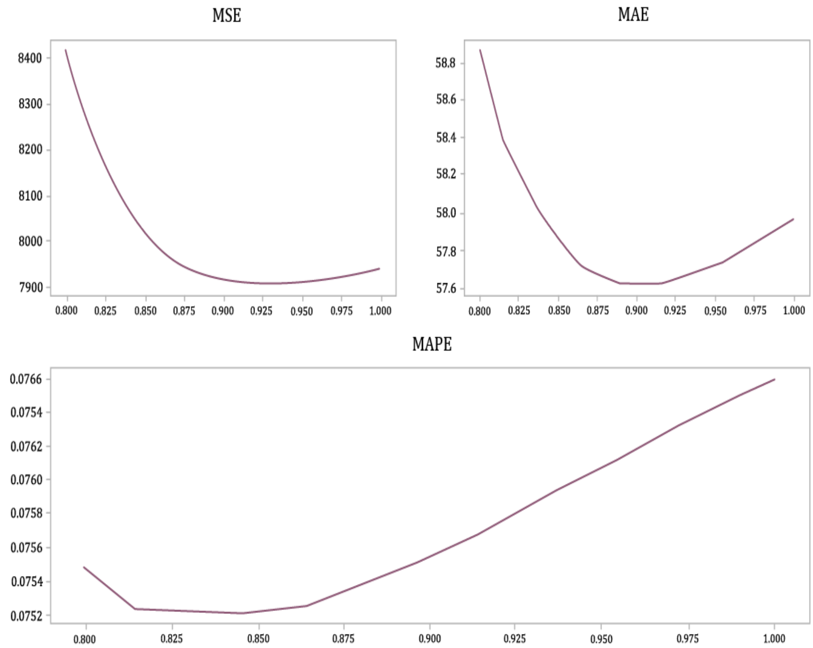
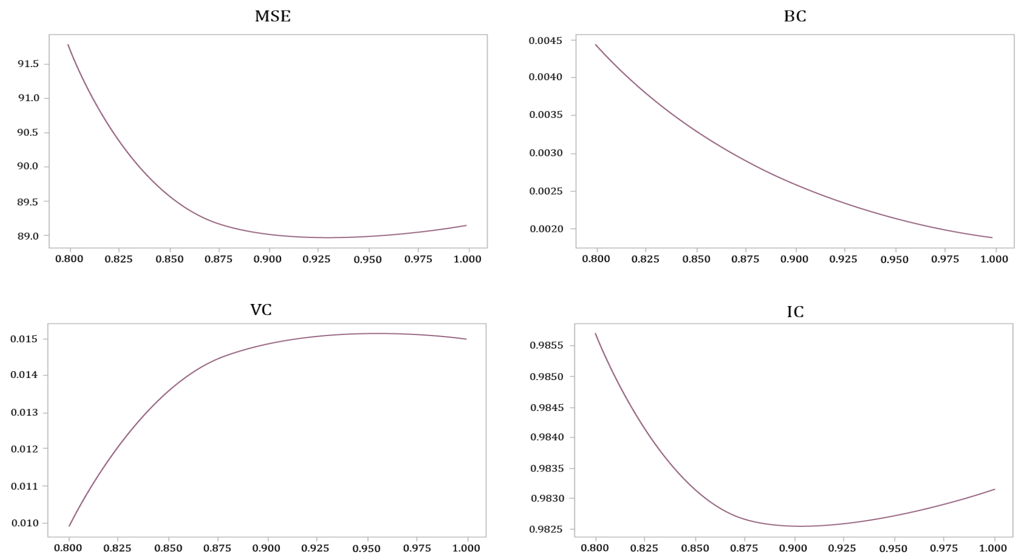


Figure 2

Mean squared error (MSE) of the forecasted prices as a function of the conformable parameter α. The choice α = 1 corresponds to the classical BSM model. The convexity of the error function shows the possibility of increased precision using conformable approximations.

p. 65



* MSE: Mean squared error.

*BC: Bias component.

*VC: Variance component.

*IC: Irregular component.

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