Alternative approach to explore the stability of floating bodies

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We present a simplified model of a boat to study its rotational dynamics, which is a significant criterion for the development of navigation systems. The stability of a floating body can be examined by means of rotational potential energy, which depends solely on the boat's gravity center and a point called the metacentric height. Typically, this geometric point is a function of the body's orientation in relation to the fluid surface, and the estimation of its value can often be ambiguous. This paper presents an alternative method for calculating the metacentric height using a vectorial approach, as well as a general definition of rotational potential energy applicable to this type of problem. The potential energy facilitates the determination of stable and unstable equilibrium directions as a function of the boat's relative density and orientation.

Keywords: Floating bodies; metacentric height; rotational stability.

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1. Introduction

The rotational stability of floating bodies is a crucial and widely studied problem in the field of fluid mechanics. It is of particular interest in engineering and physics, where the stability of boats and submarines is a subject of constant investigation. The problem is fundamentally defined by the center of gravity $\vec{r}_{CG}$ and the buoyancy point $\vec{r}_b$, where the weight and buoyancy force are applied, respectively. Depending on the geometry of the system and the submerged volume, a geometric point known as the metacentric height $\vec{r}_M$ can be identified. This vector is the point of intersection between the line of action of the buoyancy force and the symmetry axis of the body. The metacentric height is commonly determined through the following equation [1, 2]:

$$|\vec{r}_M| = \frac{I}{V_s}, \tag{1}$$

where I is the moment of inertia parallel to the fluid surface and V_s is the submerged volume. However, this expression is not always presented clearly in textbooks and other sources. Generally, a stable stability criterion is defined when $\vec{r}_M > \vec{r}_{CG}$, as this generates a restorative moment of the weight. In other words, it results in a minimum of the rotational potential energy of the center of gravity in relation to the metacentric height [3, 4]. Moreover, a fully geometric approach allows for the calculation of all vectors and the rotational potential energy in a more natural manner [3–5].

2. Description of the system

The model of a boat can be simplified to a triangular shape with constant density σ. This shape is defined by the following system of equations [5]:

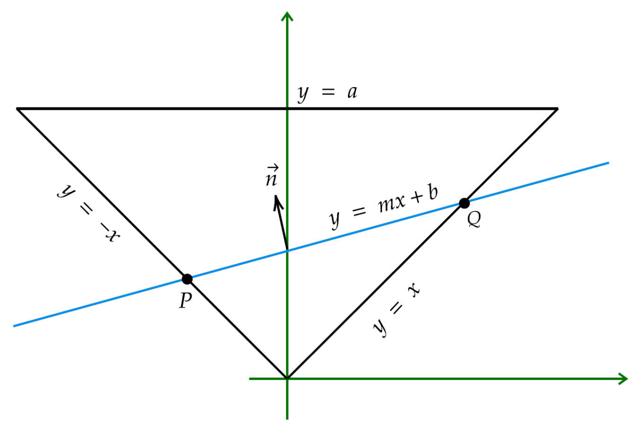

FIGURE 1. Floating body represented by a triangle shape and the fluid surface is described by a first-order polynomial.

$$\begin{aligned} y &= a, \\ y &= x, \\ y &= -x. \end{aligned} \tag{2}$$

The position of the center of gravity $\vec{r}_{CG}$ of the boat is determined by the shape of its body. It is measured from the reference points shown in Fig. 1,

$$\vec{r}_{CG} = \left(0, \frac{2}{3}a\right). \tag{3}$$

In contrast, the surface of the fluid is represented using a first-order polynomial:

$$y = mx + b. \tag{4}$$

The slope of the polynomial is limited by the body's geometry and its intersection with the fluid surface. Therefore,

$$m \leq 1 - \frac{b}{a}. \tag{5}$$

In addition, the orientation of the fluid surface is determined by the following normal vector, which specifies the stability directions,

$$\hat{n}(m) = \frac{1}{\sqrt{1+m^2}}(-m, 1). \tag{6}$$

The intersection between the fluid surface and the body is defined by the points P and Q, which are expressed as follows:

$$P = \left(\frac{-b}{1+m}, \frac{b}{1+m}\right),$$
$$Q = \left(\frac{b}{1-m}, \frac{b}{1-m}\right). \tag{7}$$

These points allows us to calculate the centroid of the submerged surface,

$$\vec{r}_b = \frac{1}{A}\int\limits_{P_x}^{Q_x}\int\limits_{|x|}^{mx+b}(x\hat{i} + y\hat{j})dxdy. \tag{8}$$

The submerged area of the body, denoted by A, is a function of the slope m and the intercept b of the polynomial that characterizes the fluid surface,

$$A = \int\limits_{P_x}^{Q_x}\int\limits_{|x|}^{mx+b} dxdy. \tag{9}$$

After performing the integral, we obtain the following expression:

$$\vec{r}_b = \frac{2}{3}b\left(\frac{m}{1-m^2}, \frac{1}{1-m^2}\right), \qquad |m| \neq 1. \tag{10}$$

The line of action of the buoyancy force is determined by a vectorial line that is generated by the buoyancy vector and the orientation of the fluid surface,

$$\vec{r}_e = \vec{r}_b + t\hat{n}, \qquad t > 0. \tag{11}$$

The metacentric vector is defined as the point at which the line of action of the buoyancy force intersects the original line of action prior to any rotation. This point is characterized by a zero x-component of the position vector, *i.e.*, $\vec{r}_R|_x = 0$:

$$t = \frac{2}{3}b\frac{\sqrt{1+m^2}}{1-m^2}. \tag{12}$$

The metacentric vector is therefore:

$$\vec{r}_M = \left(0, \frac{4}{3}b\frac{1}{1-m^2}\right). \tag{13}$$

The condition $\vec{r}_M > \vec{r}_{CG}$ defines the stability of a floating body in terms of the concept of potential energy. Specifically, if the metacentric vector $\vec{r}_M$ is above the center of gravity vector $\vec{r}_{CG}$, then the body is stable and has a lower potential energy, since any disturbance will cause a restoring moment that returns the body to its original position. On the other hand, if $\vec{r}_M$ is below $\vec{r}_{CG}$, the body is unstable and has a higher potential energy, since any disturbance will cause a destabilizing moment that will tend to overturn the body.

3. Rotational potential energy

The weight of the boat generates a net torque that can be calculated from the metacentric height [6,7]. As a reminder, the work associated with the torque can be expressed as:

$$W = \int\limits_0^\theta \tau(\theta')d\theta'. \tag{14}$$

Using the common definition of torque, we can express the potential energy of a floating body in terms of its orientation and the forces acting upon it. Specifically, the potential energy can be written as:

$$U = |\vec{r}_R||\vec{w}| - |\vec{r}_R||\vec{w}|cos(\theta). \tag{15}$$

Here, $\vec{r}_R = \vec{r}_{CG} - \vec{r}_M$ as shown in Fig. 2, represents the relative vector between the center of gravity and the metacentric height. Note that the second term in the above expression is the dot product between the relative vector and the weight (which is antiparallel to the surface). Therefore,

$$U = |\vec{w}|(|\vec{r}_R| - \vec{r}_R \cdot (-\vec{n})). \tag{16}$$

We define a rotational potential energy per unit force and length as $U(m, b/a) := U/(aw)$, which depends only on the fluid surface. In general, if the potential energy func-

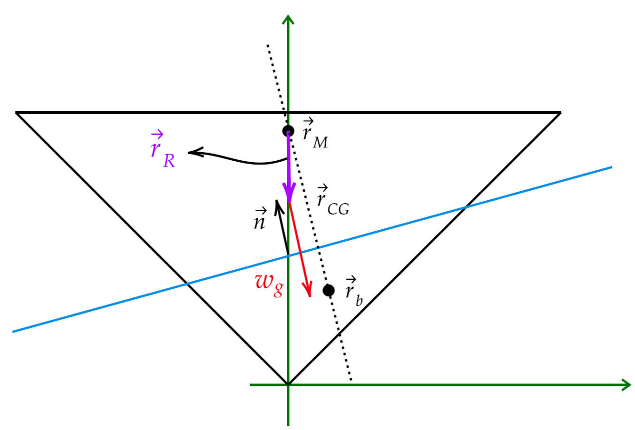

FIGURE 2. The geometry associated with the torque exerted by the weight of the boat with respect to the metacentric vector is determined by the relative position of the center of gravity and the metacentric height, as well as the orientation of the body.

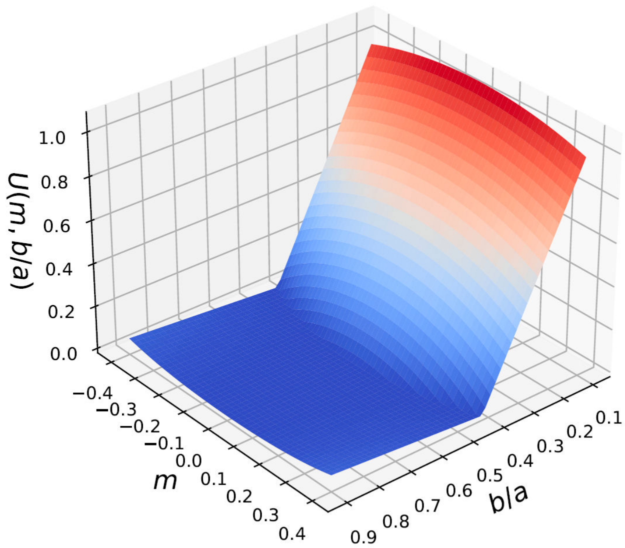

FIGURE 3. Rotational potential energy of the system associated to the metacentric height. Three stability regions are distinguished: an unstable rotational region defined by $0.1 \leq b/a \leq 0.4$, a transition region defined by $0.4 \leq b/a \leq 0.5$, and a stable region defined by $0.5 \leq b/a \leq 0.9$.

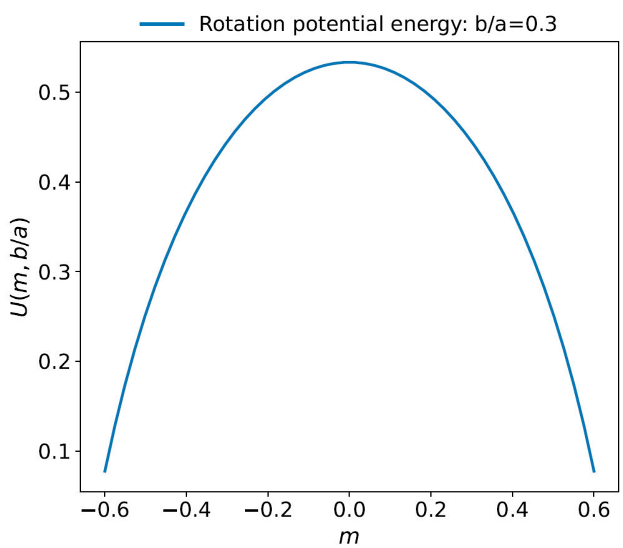

FIGURE 4. Rotational potential energy of the system for $b/a = 0.3$. This configuration corresponds to an unstable equilibrium point, as indicated by the absence of a local minimum in the potential energy curve.

tion has a local minimum, the body will remain in stable static equilibrium, while an absence of a local minimum implies an unstable equilibrium. Specifically, for $\vec{r}_R \cdot (-\vec{n}) > 0$ the equilibrium is stable due to the condition $\vec{r}_M > \vec{r}_{CG}$, which is the canonical requirement for stability. Equation (17) allows us to study the mechanical behavior of the system for different relative densities, *i.e.*, $0 < b/a < 1$.

$$U(m, b/a) = \left|\frac{2}{3}\left(1 - \frac{2b/a}{1-m^2}\right)\right| - \frac{2}{3}\left(1 - \frac{2b/a}{1-m^2}\right)\frac{1}{\sqrt{1+m^2}}. \qquad (17)$$

The rotational potential energy for relative densities $0.1 < b/a < 0.9$ is presented in Fig. 3. The system exhibits three distinct stability regions. Region I, where $0.1 \leq b/a < 0.4$, is characterized by an unstable equilibrium with a local maximum. Region II, where $0.4 \leq b/a < 0.5$, is a transition region with asymmetric stability points ($m \neq 0$). Finally, region III, where $0.5 \leq b/a < 0.9$, is a stable equilibrium region with a local minimum. Figure 4 illustrates an unstable

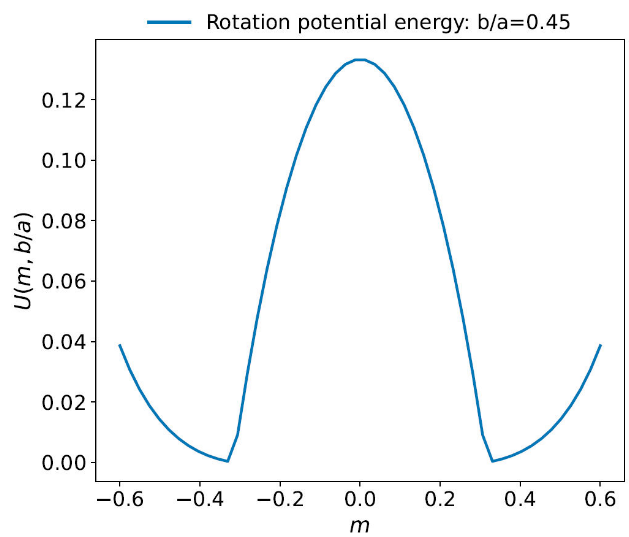

FIGURE 5. Rotational potential energy of the system for $b/a = 0.4$. This configuration corresponds to a transition equilibrium point with an asymmetric local minimum.

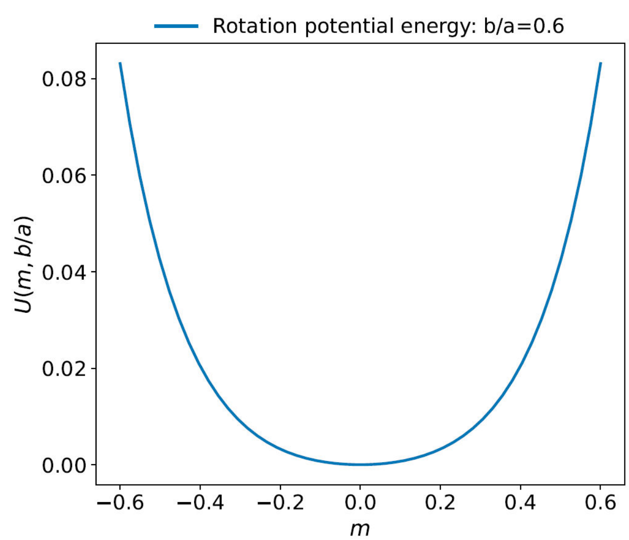

FIGURE 6. The rotational potential energy of the system is calculated for a ratio of $b/a = 0.6$. This specific configuration is indicative of a stable equilibrium point.

equilibrium point in region I, specifically at $b/a = 0.3$. The local maximum at this point is symmetric around $m = 0$.

Figure 5 shows a point of the transition region with $b/a = 0.45$, where the local minimum is roughly $|m| = 0.330$. Furthermore, the system tends to settle in the local minimum at this relative density value since the symmetry axis ($m = 0$) is an unstable point.

Figure 6 shows a stable point for the region III, where $b/a = 0.6$. The local minimum is symmetric around $m = 0$.

4. Conclusions

This communication aims to investigate the approximate behavior of the rotational stability of a floating body, which is modeled using a simplified geometry. First, we parameterize the fluid surface with a first-order polynomial and calculate the centroid of the submerged volume. Second, we utilize the vectorial expression of the straight line to determine the metacentric height of this system. This vector facilitates the calculation of the torque of the system's weight with respect to the metacentric height, and consequently, the potential energy. Third, the results reveal three stability regions: an unstable region where the relative density is $0.1 \leq b/a < 0.4$, a transition region with asymmetric stability points where $0.4 \leq b/a < 0.5$, and a stable region where $0.5 \leq b/a < 0.9$.

By using a triangular cross-section model for a ship, we can derive simple expressions to analyze the rotational potential energy function of the system. This energetic method is highly versatile and can also be extended to more complex cross-section geometries, such as a parabolic profile represented by the equation $y = x^2$. In such cases, readers can easily determine the crossing points P and Q, which are as follows:

$$P = \left(\frac{m}{2}(1-D), \frac{m^2}{2}(1-D)+b\right),$$
$$Q = \left(\frac{m}{2}(1+D), \frac{m^2}{2}(1+D)+b\right), \qquad (18)$$

where $D = \sqrt{(1+4b)/m^2}$. To estimate the rotational potential energy, the centroid and metacentric height must be calculated using the points described in this communication. For the parabolic profile, these values are more difficult to calculate, thereby, more complex shapes may require the use of numerical methods to establish the stability criteria.

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